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CHAPTER 14

# Binary black-hole mergers as hydrogen-like gravitational molecules

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## Abstract

We develop a quantum-geometrical model of binary black-hole coalescence within the Nexus Paradigm (NP), in which black holes are represented by quantized Ricci-soliton states with eigen-radii $r_n = n^2 GM/c^2$. Binary configurations are formulated as coupled two-centre gravitational eigenstates whose evolution is described on an adiabatic energy surface. Applying a semiclassical orbital correspondence to the NP spectrum gives $v_n = c/n$ and $f_n = c^3/(\pi GMn^3)$. We propose $n = 2$ as the lowest admissible massive terminal state of the binary sequence, yielding the zeroth-order prediction $f_2^{NP} = \frac{c^3}{8\pi GM}$, which can be tested directly using independently inferred source-frame masses and gravitational-wave peak frequencies. Departures from this relation are attributed to independently calculable relativistic, finite-mass-ratio, spin-orbit and eccentricity corrections. The model possesses a semiclassical correspondence limit at $n \gg 1$, whereas the strongly discrete low-$n$ spectrum provides an additional observational test near merger. Reconciling this discreteness with smooth late-time waveforms requires sufficiently overlapping radiative transitions, providing a quantitative constraint on NP transition widths and matrix elements. The resulting framework therefore identifies two principal tests of the proposed quantum structure of black-hole coalescence: the $n = 2$ mass-frequency relation and possible signatures of low-$n$ spectral discreteness in gravitational-wave signals.

**Keywords:** Nexus Paradigm; quantum gravity; binary black-hole mergers; gravitational waves

# Binary Black-Hole Mergers as Hydrogen-like Gravitational Molecules

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**Abstract**

We develop a quantum-geometrical model of binary black-hole coalescence within the Nexus Paradigm (NP), in which black holes are represented by quantized Ricci-soliton states with eigen-radii $r_n = n^2 GM/c^2$. Binary configurations are formulated as coupled two-centre gravitational eigenstates whose evolution is described on an adiabatic energy surface. Applying a semiclassical orbital correspondence to the NP spectrum gives $v_n = c/n$ and $f_n = c^3/\left(\pi GMn^3\right)$. We propose $n = 2$ as the lowest admissible massive terminal state of the binary sequence, yielding the zeroth-order prediction $f_2^{NP} = \frac{c^3}{8\pi GM}$, which can be tested directly using independently inferred source-frame masses and gravitational-wave peak frequencies. Departures from this relation are attributed to independently calculable relativistic, finite-mass-ratio, spin-orbit and eccentricity corrections. The model possesses a semiclassical correspondence limit at $n \gg 1$, whereas the strongly discrete low-$n$ spectrum provides an additional observational test near merger. Reconciling this discreteness with smooth late-time waveforms requires sufficiently overlapping radiative transitions, providing a quantitative constraint on NP transition widths and matrix elements. The resulting framework therefore identifies two principal tests of the proposed quantum structure of black-hole coalescence: the $n = 2$ mass-frequency relation and possible signatures of low-$n$ spectral discreteness in gravitational-wave signals.

# 1 Introduction

The detection of gravitational waves (GWs) from compact-binary coalescences has opened a direct observational window onto the dynamical strong-field regime of gravity [1–6]. Beginning with GW150914, observations by the LIGO–Virgo–KAGRA (LVK) collaborations have established a growing population of binary black-hole (BBH), binary neutron-star (BNS), and neutron-star–black-hole (NSBH) mergers spanning a broad range of masses, mass ratios, spins, and astrophysical environments [7–10]. These observations provide increasingly precise tests of General Relativity (GR) under conditions inaccessible to Solar-System or laboratory experiments.

Current gravitational-wave observations remain broadly consistent with GR. Tests based on waveform residuals, inspiral–merger–ringdown consistency, parameterized post-Newtonian (PN) deformations, modified gravitational-wave propagation, black-hole spectroscopy, and searches for post-merger structure have so far revealed no statistically significant evidence for departures from GR [18–23]. Contemporary catalogue-level analyses likewise constrain possible deviations in gravitational-wave generation, propagation, polarization and remnant properties.

Nevertheless, the successful description of compact-binary dynamics by classical GR does not resolve the more fundamental problem of reconciling gravitation with quantum mechanics. Questions concerning the microscopic structure of spacetime, the localization and quantization of gravitational energy, and the nature of black-hole states continue to motivate quantum-gravity research. Binary black-hole mergers are especially interesting in this context because their late evolution probes highly dynamical spacetime on length scales comparable to the gravitational radius,

$$r_g = \frac{GM}{c^2}, \tag{1}$$

where $M$ denotes the relevant gravitating mass.

The purpose of the present work is not to identify a failure of GR. Instead, we investigate whether the Nexus Paradigm (NP) [11–13] provides a distinct quantum-geometrical description of binary black-hole coalescence and, crucially, whether that description leads to independently testable gravitational-wave predictions.

## 1.1 Gravitational-wave tests of strong-field gravity

A majority of approaches to testing gravity with compact binaries start by introducing controlled deviations from GR waveforms. The parameterized post-Einsteinian (ppE) framework provides a systematic means of incorporating possible departures from GR into waveform amplitudes and phases [14,15]. Subsequent developments extended parameterized approaches to more general signal morphologies and strong-field phenomena [16,17].

The LVK collaborations have implemented related methods in systematic tests of GR using successive gravitational-wave catalogues [18–21]. These analyses include residual tests, inspiral–merger–ringdown consistency tests, parameterized modifications of PN coefficients, polarization tests, modified-dispersion constraints, and tests of the properties of merger remnants. No statistically significant violation of GR has been identified in these analyses [18–21]. The current GWTC-3 testing program likewise reports consistency with GR across residual, waveform-generation, propagation, polarization, ringdown and echo tests.

Parameterized approaches are particularly powerful because they allow observations to constrain broad classes of departures without requiring a complete alternative theory. Their interpretation, however, generally concerns deviations from an underlying GR waveform and may involve degeneracies between beyond-GR effects and higher-order astrophysical or waveform parameters [14,15,18–21]. A quantum-gravity model that predicts a specific strong-field spectral or termination scale can therefore provide a complementary class of test.

## 1.2 Propagation, ringdown and near-horizon tests

Gravitational waves also provide tests of fundamental physics through their propagation over astrophysical and cosmological distances. Modified dispersion relations, Lorentz-violating terms, or a finite graviton mass can produce frequency-dependent modifications of the observed waveform [22,23]. Such effects have consequently been constrained using gravitational-wave observations [22,23]. Recent analyses continue to improve modified-dispersion tests while finding no evidence for a violation of GR propagation.

A complementary approach examines the post-merger remnant. In GR, the ringdown of a perturbed black hole is characterized by a spectrum of Kerr quasi-normal modes (QNMs), providing the basis for black-hole spectroscopy and tests of the no-hair hypothesis [24–27]. Increasingly sophisticated analyses investigate the consistency of observed ringdown frequencies and damping times with those expected from Kerr black holes.

Quantum-gravity-inspired models in which the near-horizon region differs from the classical GR description have also motivated searches for late-time gravitational-wave echoes and other post-merger structures [27–34]. Existing observations have not established statistically compelling evidence for such signals, but they provide increasingly restrictive constraints on exotic compact objects and macroscopic modifications of the near-horizon geometry [27–34].

Together with multi-event and hierarchical inference techniques, these approaches illustrate the growing ability of gravitational-wave astronomy to constrain strong-field alternatives to GR. The present work addresses a different but complementary possibility: that quantum geometry manifests itself through a **discrete spectrum of gravitational states** governing the compact binary and its terminal configuration.

## 1.3 The Nexus Paradigm and quantized gravitational states

In the NP, we initiate the quantization procedure by considering a finite patch of Minkowski space equipped with a non-degenerate symmetric bilinear form on the tangent space. We adopt a local coordinate system to circumvent the necessity of an origin and to avoid point-like events, which are sources of divergences in quantum field theory. This local coordinate formulation renders Minkowski space a displacement vector space. The inner product is consequently given by

$$\begin{aligned} \Delta s^2 = \Delta x^\mu \Delta x_\mu = \ & \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 \\ = (A\Delta x + B\Delta y + C\Delta z + icD\Delta t)&(A\Delta x + B\Delta y + C\Delta z + icD\Delta t). \end{aligned} \tag{2}$$

Upon multiplication of the right-hand side, cancellation of all cross terms such as $\Delta x \Delta y$ requires

$$AB + BA = 0, \ \cdots \quad \text{and} \quad A^2 = B^2 = \cdots = 1. \tag{3}$$

These conditions generate a Clifford algebra, implying that the coefficients $(A,B,C,D)$ must be matrices, specifically the Dirac gamma matrices. These matrices are square roots of the Minkowski metric

$$\{\gamma^\mu, \gamma^\nu\} = 2\mathbf{I_4}\eta^{\mu\nu}. \tag{4}$$

Thus the displacement vectors $\Delta x^\mu = a\gamma^\mu$ reside in Clifford space $Cl_{1,3}(R)_C$ implying an intrinsic quantized spin and can be perceived as Dirac 4-vector matrices analogous to the Pauli vector matrices in three dimensions. They are also quantized wave packets or pulses of space-time that can be expressed as Fourier integrals as follows

$$\Delta x_n^\mu = \frac{2r_{HS}}{n\pi}\gamma^\mu \int_{-\infty}^{\infty} \mathrm{sinc}(kx) e^{ikx} dk \ = \gamma^\mu \int_{-\infty}^{\infty} a_{nk}\varphi_{(nk)} dk, \tag{5}$$

where $\sum_{k=-\infty}^{k=+\infty} a_{nk}$ , $r_{HS}$ is the Hubble radius, $\varphi_{(nk)} = \mathrm{sinc}(kx)e^{ikx}$ are Bloch energy eigenstate functions in which the four wave vectors assume the following quantized values

$$k^{\mu} = \frac{n\pi}{r_{HS}^{\mu}} \qquad n = \pm 1, \pm 2 \ldots 10^{60}. \tag{6}$$

We impose a high-energy cutoff at the Planck four-length, as no measurement can be obtained below this scale without black hole formation, and a low-energy cutoff at the Hubble four-radius, since no information can be retrieved beyond the cosmic horizon. The $10^{60}$ states arise from the ratio of Hubble four radius to the Planck four length. The Bloch functions in each eigenstate of space-time generate an infinite Bravais four lattice. The conjugate momentum for the displacement vectors is

$$\begin{aligned} \Delta p_n^{\mu} &= \frac{2np_1}{\pi}\gamma^{\mu}\int_{-\infty}^{\infty}\varphi(kx)dx \\ &= \gamma^{\mu}\int_{-\infty}^{\infty} c_n\varphi(kx)dx, \end{aligned} \tag{7}$$

where $p_1$ is the four-momentum of the ground state.

The wave packet may be interpreted as a particle of four-space and can be envisioned as enveloping a spherically symmetric concentration of an excited energy state of the quantum vacuum.

From [11-13], the wave packet can be considered as a compact Einstein manifold or a trivial Ricci soliton of positive Ricci curvature expressed in the form

$$G_{(nk)\mu\nu} = n^2\Lambda g_{(n,k)\mu\nu} = n^2\kappa\rho_{\Lambda} g_{(n,k)\mu\nu}. \tag{8}$$

The semi classical solution of Equation (8) is expressed in the form

$$ds^2 = -\left(1-\left(\frac{2}{n^2}\right)\right)c^2dt^2 + \left(1-\left(\frac{2}{n^2}\right)\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{9}$$

where $n^2 = \frac{c^2 r_n}{GM} = \frac{c^2}{v^2}$. The full quantum formulation is derived from expressing the Ricci flow in terms of the Bloch wave functions

$$i\partial_t\{\gamma_{\mu},\gamma_{\nu}\}\varphi_{(nk)}\varphi_{(nk)} = -\frac{cr_{HS}}{12\pi^2}\{\gamma_{\mu},\gamma_{\nu}\}\nabla^{(n-1)\mu}\varphi_{(nk)}\nabla_{(n+1)\nu}\varphi_{(nk)} + n^2H_0\{\gamma_{\mu},\gamma_{\nu}\}\varphi_{(nk)}\gamma_{\nu}\varphi_{(nk)}, \tag{10}$$

where

$$\nabla^{(n-1)\mu} = \partial^{n\mu} - ik^{1\mu},\ \nabla_{(n+1)\nu} = \partial_{(n\nu)} + ik_{1\nu} \quad \text{and} \quad \frac{1}{3}cr_{HS}\Lambda = H_0.$$

## 1.4 Scope and objectives of this work

The present paper develops this framework in three stages. First, we formulate binary black holes as coupled two-centre Ricci-soliton eigenstates and establish the corresponding adiabatic gravitational energy structure. Second, we examine gravitational-wave emission and the relationship between the discrete NP spectrum, its large-$n$ semiclassical limit, and the strongly discrete terminal regime. Third, we confront the $n = 2$ hypothesis with gravitational-wave observations through the independently predicted mass-frequency relation $Mf_{peak}^{(0)} = \frac{c^3}{8\pi G}$.

The purpose is not to argue that existing gravitational-wave observations contradict GR; current LVK tests continue to find consistency with GR. Rather, the objective is to determine whether the NP framework supplies an independently falsifiable quantum-geometrical description of the terminal binary state. The analysis therefore focuses on two distinct predictions, that is the zeroth-order $n = 2$ mass-frequency relation and the possible observational consequences of low-$n$ spectral discreteness. Deviations from the zeroth-order relation are treated through independently calculable relativistic, mass-ratio, spin-orbit and eccentricity corrections. This formulation separates the underlying NP quantization hypothesis from observational diagnostics and provides a direct route for testing the proposed quantum structure of black-hole coalescence with present and future gravitational-wave observations.

This immediately suggests a molecular interpretation of binary black-hole systems.

Just as two hydrogen atoms combine through overlap of electron wavefunctions to form molecular orbitals, two black-hole Ricci solitons may form a coherent two-centre spacetime eigenstate before transitioning into a single final soliton.

Table 1: Correspondence between Atomic physics and binary black holes

| **Atomic physics** | **Binary black holes** |
|---|---|
| proton centres | Ricci soliton centres |
| electron wavefunction | metric eigenfunction |
| molecular orbital | binary spacetime state |
| photon emission | gravitational-wave emission |
| atomic transition | soliton transition |

The analogy is:
The purpose of this work is to construct this correspondence quantitatively.

# 2 Quantized Ricci-Soliton Eigenstates and the Binary Molecular Construction

## 2.1 NP eigen-radius spectrum

The starting point of the Nexus Paradigm (NP) is the quantization of the spacetime geometry itself rather than the canonical quantization of a test particle moving in a prescribed Newtonian potential. In this light, the characteristic length scale of black-hole eigenstates is not derived by inserting a gravitational analogue of the Coulomb potential into the standard Schrödinger equation.

For a spherically symmetric NP gravitational eigenstate, making comparisons of the quantized metric with the corresponding Schwarzschild exterior gives the dimensionless eigenvalue relation $\frac{GM}{c^2 r_n} = \frac{1}{n^2}$, $n = 1,2,3,\ldots$, where $r_g \equiv \frac{GM}{c^2}$ is the gravitational radius associated with the total gravitating mass $M$. It then follows immediately that $r_n = n^2 r_g = n^2 \frac{GM}{c^2}$ . This quantized radial expression constitutes the fundamental radial quantization condition employed in this work. This spectrum should therefore not be interpreted as the Bohr spectrum of a material particle orbiting in a potential $V(r)$. This distinction is important. In the conventional hydrogen atom the characteristic Bohr length follows from the combination $a_0 = \frac{\hbar^2}{m\kappa}$ where $m$ is the particle mass and $\kappa$ is the Coulomb coupling. No corresponding identification of an effective graviton mass $m_G$, gravitational Coulomb coupling, or effective Planck constant is required to obtain the quantized radii. The fundamental quantized object in the NP construction is instead the gravitational geometry.

## 2.2 Velocity and frequency associated with the eigen-radius

Once the NP eigen-radius has been specified, a useful semiclassical velocity scale can be associated with it through the circular-orbit relation $v_n^2 = \frac{GM}{r_n}$ or $v_n^2 = \frac{GM}{n^2 GM/c^2} = \frac{c^2}{n^2}$, and therefore $v_n = \frac{c}{n}$.

The orbital angular frequency associated with $r_n$ is

$$\Omega_n = \frac{v_n}{r_n} = \sqrt{\frac{GM}{r_n^3}}, \tag{11}$$

which, can be also expressed as

$$\Omega_n = \frac{c^3}{GMn^3}. \tag{12}$$

For a dominant quadrupolar gravitational-wave mode whose frequency is twice the orbital frequency,

$$f_n = \frac{\Omega_n}{\pi}, \tag{13}$$

so that

$$f_n = \frac{c^3}{\pi GMn^3}. \tag{14}$$

Equations (9)–(12) are the kinematic consequences of the single eigen-radius spectrum, Eq. (6) and become important when confronting the theory with gravitational-wave observations.

## 2.3 The $n = 2$ terminal state

The binary-merger model considered here associates the terminal bound configuration with the lowest admissible massive black-hole state,

$$n = 2. \tag{15}$$

The corresponding characteristic radius and velocity scale are therefore

$$r_2 = 4r_g = \frac{4GM}{c^2}, \tag{16}$$

and

$$v_2 = \frac{c}{2}. \tag{17}$$

Equation (12) then gives the zeroth-order terminal gravitational-wave frequency

$$f_2^{NP} = \frac{c^3}{8\pi GM}. \tag{18}$$

Equivalently,

$$Mf_2^{NP} = \frac{c^3}{8\pi G}. \tag{19}$$

Equation (18) and (19) are the principal zeroth-order observational prediction tested in this work. Unlike an effective quantum number reconstructed from an observed frequency or velocity, it depends only upon the independently measured source-frame total mass $M$.

Corrections due to strong-field relativistic dynamics, finite mass ratio, spin-orbit coupling and eccentricity are considered subsequently as perturbations to this zeroth-order relation. They are not incorporated by defining an observationally inferred $n_{eff}$.

## 2.4 Meaning of the hydrogen-like terminology

The term *hydrogen-like* used in this work refers to the discrete $n^2$ eigen-radius structure and to the construction of binary states from localized gravitational eigenstates. It does not imply that the NP black-hole state is an ordinary hydrogen atom with the replacement of the electromagnetic Coulomb interaction by a Newtonian gravitational potential.

To make this distinction explicit, let

$$|A,n\rangle \text{ and } |\,B,n\rangle \tag{20}$$

denote localized NP metric eigenstates associated with the two compact objects. When the objects are sufficiently separated, these states form an approximately localized basis. Their overlap is

$$S_n(R) = \langle A,n,|\,Bn\rangle\,, \tag{21}$$

where $R$ is the binary separation.

The lowest symmetric and antisymmetric two-centre combinations may then be written

$$\left|\Psi_n^{(+)}\right\rangle = \frac{|A,n\rangle + |\,B,n\rangle}{\sqrt{2\,(1+S_n)}}, \tag{22}$$

and

$$\left|\Psi_n^{(-)}\right\rangle = \frac{|A,n\rangle - |\,B,n\rangle}{\sqrt{2\,(1-S_n)}}. \tag{23}$$

These expressions are mathematically analogous to the linear-combination-of-atomic-orbitals construction used for molecular systems. In the present context, however, the states being combined represent gravitational metric eigenstates rather than electronic wavefunctions.

The corresponding two-centre Hamiltonian may be represented schematically as

$$\widehat{H}_g(R) = \widehat{H}_A + \widehat{H}_B + \widehat{H}_{int}(R), \tag{24}$$

with matrix elements

$$H_{AA} = \left\langle A \mid \widehat{H}_g \mid A \right\rangle, \; H_{AB} = \left\langle A \mid \widehat{H}_g \mid B \right\rangle. \tag{25}$$

Diagonalization in the two-state basis gives

$$E_{\pm}(R) = \frac{H_{AA} \pm H_{AB}}{1 \pm S(R)} \tag{26}$$

for identical localized states. The symmetric and antisymmetric branches therefore arise from the overlap and interaction of the two gravitational eigenstates.

This is the precise sense in which the binary is described here as a gravitational molecular system.

## 2.5 Born–Oppenheimer separation

The molecular analogy also motivates an adiabatic separation between the rapidly adjusting gravitational configuration and the comparatively slow evolution of the binary separation $R$. The total state may therefore be written schematically as

$$\Psi[g,R] = \psi_\alpha[g;R]\,\Phi_\alpha(R), \tag{27}$$

where $g$ denotes the gravitational degrees of freedom.

At fixed binary separation $R$, the gravitational state satisfies

$$\widehat{H}_g(R)\psi_\alpha[g;R] = \varepsilon_\alpha(R)\psi_\alpha[g;R]. \tag{28}$$

The resulting eigenvalue $\varepsilon_\alpha(R)$ acts as an effective potential for the slower relative binary motion,

$$\left[-\frac{\hbar^2}{2\mu}\nabla_R^2 + \varepsilon_\alpha(R)\right]\Phi_\alpha(R) = E\Phi_\alpha(R), \tag{29}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{30}$$

is the reduced mass of the binary. A large $\mu$ implies a highly localised barycentre for the BBH

The reduced mass therefore belongs to the slow relative-motion sector, rather than being identified with the mass of a hypothetical fast gravitational analogue of the electron.

## 2.6 Approach to merger

As the binary separation decreases, the overlap $S_n(R)$ between the localized gravitational eigenstates increases. Consequently, the description in terms of two approximately independent soliton states progressively gives way to a collective two-centre state.

In this work, we do not identify merger by assuming a hydrogenic radial uncertainty $\Delta r_n$ and imposing an overlap condition such as

$$R \simeq 2\Delta r_n. \tag{31}$$

Such a condition would require an explicitly derived radial wavefunction and is not implied by the eigen-radius relation (6). Rather, we formulate a criterion in terms of the adiabatic eigenvalue $\varepsilon_+(R)$. Here a metastable binary configuration corresponds to a local minimum that satisfies

$$\frac{d\varepsilon_+}{dR} = 0, \; \frac{d^2\varepsilon_+}{dR^2} > 0. \tag{32}$$

The disappearance of this minimum occurs at a critical separation $R_c$ satisfying

$$\frac{d\varepsilon_+}{dR}\Big|_{R_c} = 0,\ \frac{d^2\varepsilon_+}{dR^2}\Big|_{R_c} = 0. \tag{33}$$

Equation (33) provides a well-defined theoretical criterion for the loss of the two-centre metastable configuration. Determining $R_c$ quantitatively requires an explicit NP two-soliton interaction Hamiltonian and is therefore distinguished from the independently established eigen-radius $r_2 = 4r_g$.

In this view, $r_2 = 4r_g$ is the characteristic radius of the $n = 2$ NP eigenstate, while the dynamical relation between this eigen-radius and the precise binary coalescence separation requires the two-centre calculation described above.

# 3 Two-Centre Gravitational Eigenstates and Molecular-State Formation

Consider two compact NP gravitational eigenstates centred on $A$ and $B$, separated by a coordinate distance $R$. For sufficiently large $R$, the gravitational configuration may be approximated by two localized states,

$$|A,n\rangle\,,\ |\,B,n\rangle\,, \tag{34}$$

each characterized by the NP eigen-radius

$$r_n = n^2\frac{GM}{c^2}. \tag{35}$$

The degree of interaction between the two configurations is characterized by the overlap

$$S_n(R) = \langle A,n,|\,Bn\rangle\,. \tag{36}$$

For large separation,

$$S_n(R) \to 0,\ R \to \infty, \tag{37}$$

and the two compact objects behave approximately as independent gravitational eigenstates. As the binary inspirals, $R$ d ecreases and the overlap becomes increasingly important.

The gravitational Hamiltonian in the localized basis can be represented by

$$H(R) = \begin{pmatrix} H_{AA} & H_{AB}\,H_{BA} & H_{BB}\end{pmatrix}, \tag{38}$$

where

$$H_{ij} = \left\langle i,n,|\,\widehat{H}_g(R)\,|\,jn\right\rangle. \tag{39}$$

Because the localized states need not be orthogonal, the corresponding overlap matrix is

$$S(R) = \begin{pmatrix} 1 & S_n\,S_n^* & 1\end{pmatrix}. \tag{40}$$

The allowed collective states are therefore obtained from the generalized eigenvalue equation

$$det\,[H(R) - E(R)S(R)] = 0. \tag{41}$$

For an approximately symmetric binary for which

$$H_{AA} = H_{BB} = H_0,\ H_{AB} = H_{BA} = K, \tag{42}$$

and with real $S_n$, Eq. (38) gives

$$E_\pm(R) = \frac{H_0 \pm K(R)}{1 \pm S_n(R)}. \tag{43}$$

The associated normalized states are

$$\left|\Psi_n^{(+)}\right\rangle = \frac{|A,n\rangle + |\,B,n\rangle}{\sqrt{2\,[1+S_n(R)]}}, \tag{44}$$

and

$$\left|\Psi_n^{(-)}\right\rangle = \frac{|A,n\rangle - |\,B,n\rangle}{\sqrt{2\,[1-S_n(R)]}}. \tag{45}$$

These states are analogous to the symmetric and antisymmetric combinations encountered in molecular quantum mechanics. Their physical interpretation here, however, is gravitational. That is they represent collective configurations of the two interacting NP metric eigenstates.

## 3.1 Bonding and antibonding branches

From Equation (40),

$$E_+ - E_- = \frac{2\,[K - H_0 S_n]}{1 - S_n^2}. \tag{46}$$

Consequently,

$$E_+ < E_- \iff K(R) < H_0 S_n(R), \tag{47}$$

provided $|\,S_n\,| < 1$.

Thus the symmetric state constitutes the lower-energy or "bonding" branch only when the NP interaction matrix elements satisfy Equation (44).

The terminology *bonding* and *antibonding* refers to the lower- and higher-energy branches of the two-centre gravitational eigenvalue problem, rather than as an assumption imported directly from the hydrogen molecular ion.

## 3.2 Adiabatic evolution during inspiral

During the early inspiral, the binary separation evolves on a timescale that may be long compared with the adjustment of the gravitational configuration. Under this adiabatic approximation, the two-centre eigenvalue

$$\varepsilon_\alpha(R) \tag{48}$$

defines an effective energy surface for the slower relative motion of the compact objects.

The binary may therefore evolve approximately along one of the instantaneous gravitational eigenbranches,

$$\widehat{H}_g(R)\,|\,\Psi_\alpha(R)\Big\rangle = \varepsilon_\alpha(R)\,|\,\Psi_\alpha(R)\Big\rangle. \tag{49}$$

As gravitational radiation removes orbital energy and angular momentum, $R$ decreases and both the overlap $S_n(R)$ and interaction matrix element $K(R)$ evolve. The initially localized description consequently becomes progressively less appropriate, while the collective two-centre description becomes increasingly important.

This provides the gravitational analogue of molecular-state formation used in the present model.

## 3.3 Transition from the binary to the terminal state

It is important to note that the collective two-centre state is not the separation at which merger occurs. In particular, the NP eigen-radius $r_2 = 4r_g$ must not automatically be identified with the centre-to-centre binary separation $R$.These are physically distinct quantities. That is $r_2$ characterizes the $n = 2$ NP gravitational eigenstate, whereas $R$ describes the relative separation of the two compact objects.

A dynamical transition is associated with the loss of a metastable minimum of the relevant adiabatic energy branch. If $\varepsilon_+(R)$ denotes that branch, a stable or metastable configuration requires $\frac{d\varepsilon_+}{dR} = 0,\ \frac{d^2\varepsilon_+}{dR^2} > 0$ . At a critical separation $R_c$, disappearance of this minimum would satisfy $\frac{d\varepsilon_+}{dR}\big|_{R_c} = 0,\ \frac{d^2\varepsilon_+}{dR^2}\big|_{R_c} = 0$ . Determination of $R_c$ therefore requires the explicit two-soliton interaction $K(R)$, overlap $S_n(R)$, and the resulting NP adiabatic energy surface. The robust quantity carried forward into the observational analysis is instead the terminal $n = 2$ eigenstate, with the following characteristics

$$r_2 = \frac{4GM}{c^2},\ v_2 = \frac{c}{2},\ f_2^{NP} = \frac{c^3}{8\pi GM}. \tag{50}$$

Perturbations of the corresponding frequency due to mass ratio, relativistic dynamics, spin-orbit coupling and eccentricity are treated separately in Section 7.

# 4 Adiabatic Energy Structure and Evolution of the Binary Gravitational Molecule

We now consider how these collective states evolve as the binary separation decreases and how this evolution is connected to gravitational-wave emission.

The molecular terminology used below refers to the mathematical analogy between coupled gravitational eigenstates and the two-centre eigenvalue problem of molecular physics. It does not imply that the binary is governed by a Coulomb-like potential or that its gravitational eigenstates are ordinary electronic orbitals.

## 4.1 Adiabatic energy branches

Let $R$ denote the instantaneous separation of the two compact objects. At fixed $R$, the gravitational sector satisfies the eigenvalue equation

$$\widehat{H}_g(R) \mid \Psi_\alpha(R)\Big\rangle = \varepsilon_\alpha(R) \mid \Psi_\alpha(R)\Big\rangle, \tag{51}$$

where $\alpha$ labels the collective gravitational branch.

For the two-state construction developed in Section 3, the symmetric and antisymmetric branches have energies

$$\varepsilon_\pm(R) = \frac{H_0(R) \pm K(R)}{1 \pm S(R)}, \tag{52}$$

where $H_0$ is the diagonal matrix element,

$$H_0(R) = \Big\langle A \mid \widehat{H}_g(R) \mid A \Big\rangle = \Big\langle B \mid \widehat{H}_g(R) \mid B \Big\rangle, \tag{53}$$

$K(R)$ is the off-diagonal interaction matrix element,

$$K(R) = \Big\langle A \mid \widehat{H}_g(R) \mid B \Big\rangle, \tag{54}$$

and

$$S(R) = \langle A \mid B \rangle \tag{55}$$

is the overlap between the localized gravitational eigenstates.

For sufficiently large separation,

$$S(R) \to 0,\ K(R) \to 0,\ R \to \infty, \tag{56}$$

so that the binary is approximately described by two independent localized gravitational states. As the separation decreases, both the overlap and interaction terms become increasingly important and the eigenstates become genuinely collective two-centre configurations.

The energy splitting between the two branches is

$$\varepsilon_+ - \varepsilon_- = \frac{2\,[K(R) - H_0(R)S(R)]}{1 - S^2(R)}. \tag{57}$$

Consequently, the symmetric state is the lower-energy branch only when

$$K(R) < H_0(R)S(R). \tag{58}$$

The interaction between two Ricci-soliton geometries need not reproduce the sign structure of the electronic $H_2^+$ problem.

## 4.2 Adiabatic inspiral along the gravitational energy surface

The two-centre gravitational eigenvalue $\varepsilon_\alpha(R)$ defines an effective energy surface for the slower relative motion of the compact objects. In the Born–Oppenheimer representation introduced in Section 2, the complete state is written schematically as

$$\Psi[g,R] = \psi_\alpha[g;R]\,\Phi_\alpha(R), \tag{59}$$

where $g$ denotes the gravitational degrees of freedom and $R$ describes the relative binary motion.

The fast gravitational sector obeys

$$\widehat{H}_g(R)\,\psi_\alpha[g;R] = \varepsilon_\alpha(R)\,\psi_\alpha[g;R], \tag{60}$$

while the slow relative motion evolves on the resulting adiabatic energy surface. Schematically,

$$\left[-\frac{\hbar_{eff}^2}{2\mu}\nabla_R^2 + \varepsilon_\alpha(R)\right]\Phi_\alpha(R) = E\Phi_\alpha(R), \tag{61}$$

with $\mu = \frac{m_1 m_2}{m_1+m_2}$ the reduced mass of the binary.

Thus the reduced mass governs the relative binary motion, rather than serving as the mass parameter of a hypothetical gravitational analogue of the electron.

During the early and intermediate inspiral, gravitational-wave emission gradually removes orbital energy and angular momentum. If the gravitational configuration adjusts more rapidly than the secular evolution of $R$, the system may approximately follow an instantaneous eigenbranch $\varepsilon_\alpha(R)$. This constitutes the adiabatic regime of the gravitational molecular description.

The validity of this approximation requires non-adiabatic coupling between distinct branches to remain sufficiently small. Schematically,

$$\left|\frac{\left\langle \Psi_\beta \mid \hat{\dot{H}}_g \mid \Psi_\alpha \right\rangle}{\left[\varepsilon_\beta(R) - \varepsilon_\alpha(R)\right]^2}\right| \ll 1,\ \beta \neq \alpha, \tag{62}$$

with the appropriate NP action scale understood.

## 4.3 Loss of the metastable two-centre configuration

As the binary evolves toward stronger interaction, the description in terms of two weakly coupled localized gravitational states eventually ceases to be adequate. The transition to merger should therefore be characterized dynamically through the structure of the adiabatic energy surface rather than through an assumed hydrogenic radial width. Suppose the relevant branch $\varepsilon_\alpha(R)$ possesses a metastable minimum at $R = R_0$.Then $\frac{d\varepsilon_\alpha}{dR}\,|_{R_0} = 0,\ \frac{d^2\varepsilon_\alpha}{dR^2}\,|_{R_0} > 0$ .

As the binary loses energy and angular momentum, the effective energy surface evolves. The disappearance of the metastable two-centre configuration may be characterized by a critical separation $R_c$ satisfying $\frac{d\varepsilon_\alpha}{dR}\big|_{R_c} = 0,\ \frac{d^2\varepsilon_\alpha}{dR^2}\big|_{R_c} = 0$ . Importantly, $R_c$ and the NP eigen-radius $r_n$ represent different physical quantities. The former is the dynamical centre-to-centre separation at which the metastable binary configuration ceases to exist, whereas $r_n = n^2\frac{GM}{c^2}$ characterizes a gravitational eigenstate. Consequently, the $n = 2$ eigen-radius, $r_2 = \frac{4GM}{c^2}$ should not by itself be identified with $R_c$. A quantitative determination of $R_c$ requires the explicit NP two-soliton interaction through $K(R)$, $S(R)$, and the resulting energy surface $\varepsilon_\alpha(R)$.

## 4.4 Gravitational-wave emission as a transition between gravitational states

The loss of energy from the binary is accompanied by gravitational radiation. Within the quantum-geometrical interpretation adopted here, a radiative transition may be represented schematically as

$$|\Psi_i\rangle \longrightarrow |\,\Psi_f\rangle + h_{GW}, \tag{63}$$

where $h_{GW}$ denotes the emitted gravitational excitation. Energy conservation requires

$$E_i - E_f = \hbar\omega_{GW}, \tag{64}$$

when the radiative field is expressed in terms of gravitational quanta.

The transition probability is governed by the coupling between the gravitational eigenstates and the radiative degrees of freedom. It should therefore be expressed through a transition matrix element,

$$\mathscr{M}_{fi} = \left\langle \Psi_f \mid \widehat{H}_{rad} \mid \Psi_i \right\rangle, \tag{65}$$

rather than by automatically identifying the gravitational-wave energy with the separation between the symmetric and antisymmetric molecular branches.

For the leading gravitational-wave channel, the radiative coupling is quadrupolar. The relevant transition amplitude consequently contains a mass-quadrupole matrix element of the form

$$\mathscr{M}_{fi} \propto \left\langle \Psi_f \mid \widehat{Q}_{ij} \mid \Psi_i \right\rangle, \tag{66}$$

where $\widehat{Q}_{ij}$ is the quadrupole operator.

The associated orbital-angular-momentum selection rule is

$$\Delta l = 0, \pm 2, \tag{67}$$

subject to the angular-momentum and parity conditions of the states involved.

The analogy is therefore with a quadrupolar transition between gravitational eigenstates, rather than with an electric-dipole atomic transition.

## 4.5 Discrete eigenstates and the continuous gravitational-wave chirp

If $n = 1, 2, 3, \ldots$ then transitions between individual stationary eigenstates are discrete, whereas observed compact-binary signals exhibit a continuously evolving gravitational-wave chirp. This distinction does not require the quantum number itself to become continuous. A time-dependent binary may instead occupy a coherent superposition of NP eigenstates,

$$|\Psi(t)\rangle = \sum_n c_n(t) \mid n \Big\rangle, \tag{68}$$

where the amplitudes $c_n(t)$ evolve as the system radiates.

For a stationary eigenbasis with energies $E_n$, the state may be expressed as

$$|\Psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} \mid n \Big\rangle. \tag{69}$$

The observable quadrupole moment is then

$$Q_{ij}(t) = \left\langle \Psi(t) \mid \widehat{Q}_{ij} \mid \Psi(t) \right\rangle, \tag{70}$$

giving

$$Q_{ij}(t) = \sum_{m,n} c_m^*(t) c_n(t) Q_{ij}^{mn} e^{-i(E_n - E_m)t/\hbar}, \tag{71}$$

where

$$Q_{ij}^{mn} = \left\langle m \mid \widehat{Q}_{ij} \mid n \right\rangle. \tag{72}$$

The radiative signal therefore depends upon the coherent evolution and interference of multiple transition amplitudes rather than upon a single continuously varying quantum number.

At large $n$, the fractional separation between neighboring eigen-radii decreases,

$$\frac{r_{n+1} - r_n}{r_n} = \frac{2n+1}{n^2} \simeq \frac{2}{n}, \; n \gg 1, \tag{73}$$

so that the spectrum becomes increasingly dense in a relative sense. This provides a natural semiclassical limit during the early inspiral.

Near the terminal state, however, $n$ is small and the discrete structure becomes much more pronounced. Consequently, a smooth waveform near merger requires either overlapping transition frequencies, finite state widths, coherent superposition, or another dynamical broadening mechanism.

If $\Gamma_n$ denotes the characteristic linewidth of a transition, neighboring transitions become unresolved when approximately

$$\Gamma_n \gtrsim |\omega_{n,n-1} - \omega_{n-1,n-2}| . \tag{74}$$

A first-principles calculation of the transition rates and linewidths from the NP radiative Hamiltonian is required to determine whether this condition is satisfied near merger.

### 4.6 Transition to the $n = 2$ terminal state

Our present model identifies $n = 2$ as the lowest admissible massive terminal black-hole eigenstate. Its characteristic eigen-radius is $r_2 = 4r_g = \frac{4GM}{c^2}$ with $v_2 = \frac{c}{2}$ and a the corresponding zeroth-order gravitational-wave frequency is $f_2^{NP} = \frac{c^3}{8\pi GM}$ or equivalently,

$$Mf_2^{NP} = \frac{c^3}{8\pi G}. \tag{75}$$

Equation (75) constitutes the principal zeroth-order mass-frequency relation carried forward to comparison with gravitational-wave observations.

An observed departure from Equation (75) is defined as the residual

$$\delta_f^{obs} = \frac{f_{peak}^{obs} - f_2^{NP}}{f_2^{NP}}. \tag{76}$$

The theoretical task is then to determine whether independently calculated corrections associated with strong-field relativistic dynamics, finite mass ratio, spin-orbit coupling and eccentricity can account for this residual. Schematically,

$$f_{peak}^{NP} = f_2^{NP} [1 + \delta_{rel} + \delta_q + \delta_{SO} + \delta_e + \cdots] . \tag{77}$$

These corrections must be obtained independently of $f_{peak}^{obs}$. The spin-orbit contribution, in particular, is signed and depends upon the projections of the component spins relative to the orbital angular momentum, as developed in Section 7.

The $n = 2$ state provides a sharply defined zeroth-order hypothesis, while departures from it constitute quantitative tests of the perturbative extension of the NP binary model.

## 5 The $n = 2$ Terminal Black-Hole State

The discrete eigen-radius spectrum $r_n = n^2 r_g$, $r_g = \frac{GM}{c^2}$, $n = 1, 2, 3, \ldots$ derived in Section 2, provides the fundamental NP quantization condition used in the present binary model. The purpose of this section is to identify the physical status of the lowest members of this spectrum and to formulate the $n = 2$ terminal-state hypothesis in a form that can be tested independently against gravitational-wave observations.

### 5.1 Physical status of the $n = 1$ state

The formal lowest member of the eigen-radius spectrum is

$$r_1 = r_g = \frac{GM}{c^2}. \tag{78}$$

Under the semiclassical velocity mapping of Equation (10), this gives $v_1 = c$

Equation (78) is interpreted here as marking the boundary of applicability of the massive timelike orbital interpretation. A massive compact object cannot possess a local physical orbital speed equal to $c$. Consequently, the $n = 1$ state cannot be interpreted as an ordinary massive binary orbital state within this semiclassical correspondence. This distinction is particularly important because $r_1 = r_g = \frac{1}{2}R_s$
where $R_s = \frac{2GM}{c^2}$ is the Schwarzschild radius associated with the same mass parameter in General Relativity. Thus $r_1$ lies inside the conventional Schwarzschild horizon

## 5.2 The $n = 2$ terminal-state hypothesis

If $n = 1$ does not admit the massive timelike orbital interpretation used for the binary system, the lowest member of the spectrum for which

$$v_n < c \tag{79}$$

is

$$n = 2. \tag{80}$$

It is important, however, to distinguish the eigen-radius $r_2$ from the centre-to-centre separation of the binary. In particular, $R_c \neq r_2$ in general. Where $R_c$ denotes the critical binary separation at which the metastable two-centre configuration ceases to exist.

As developed in Section 4, $R_c$ is determined dynamically from the relevant adiabatic energy surface $\varepsilon_\alpha(R)$, through the conditions $\frac{d\varepsilon_\alpha}{dR}\big|_{R_c} = 0$, $\frac{d^2\varepsilon_\alpha}{dR^2}\big|_{R_c} = 0$. By contrast, $r_2$ follows from the NP eigenvalue spectrum.

## 5.3 Quantifying departures from the $n = 2$ prediction

The comparison between the zeroth-order prediction and observation can be expressed through the fractional frequency residual

$$\delta_f^{obs} = \frac{f_{peak}^{obs} - f_2^{NP}}{f_2^{NP}}. \tag{81}$$

Thus

$$\delta_f^{obs} = 0 \tag{82}$$

corresponds to exact agreement with the unperturbed $n = 2$ prediction, while non-zero values measure departures from it.

For descriptive purposes, the observed mass and peak frequency can also be mapped onto the quantity

$$n_{obs} = \left( \frac{c^3}{\pi G M f_{peak}^{obs}} \right)^{1/3}. \tag{83}$$

Using Equation (81), this may equivalently be written

$$n_{obs} = 2 \left( \frac{f_2^{NP}}{f_{peak}^{obs}} \right)^{1/3} = 2 \left( 1 + \delta_f^{obs} \right)^{-1/3}. \tag{84}$$

However, $n_{obs}$ is not an NP prediction and is not an adjustable quantum number. It is simply an observational diagnostic expressing the measured peak frequency in units of the NP spectral relation. The theoretical prediction remains the integer terminal state $n = 2$.

## 5.4 Perturbations of the terminal-state frequency

A realistic binary close to merger is not characterized solely by a spherically symmetric, non-spinning, unperturbed gravitational state. Strong-field relativistic effects, unequal component masses, spin-orbit interactions and residual eccentricity may modify the frequency associated with the terminal configuration.

These effects should be introduced as independently calculated perturbations,

$$f_{peak}^{NP} = f_2^{NP} \left[ 1 + \delta_{rel} + \delta_q + \delta_{SO} + \delta_e + \cdots \right], \tag{85}$$

where $\delta_{rel}$ represents strong-field relativistic corrections, $\delta_q$ finite-mass-ratio effects, $\delta_{SO}$ spin-orbit coupling, and $\delta_e$ eccentricity-dependent corrections.

The crucial requirement is that these quantities must be obtained from theoretical calculations and independently measured binary parameters rather than determined from $f_{peak}^{obs}$.

Thus the appropriate test is

$$\delta_f^{th} = \delta_{rel} + \delta_q + \delta_{SO} + \delta_e + \cdots \tag{86}$$

against

$$\delta_f^{obs} = \frac{f_{peak}^{obs} - f_2^{NP}}{f_2^{NP}}. \tag{87}$$

Agreement between Equations (86) and (87) would constitute a non-trivial test of the perturbative NP model, whereas disagreement would constrain or falsify the assumed correction structure.

For small perturbations, it is sometimes convenient to parameterize the theoretical shift relative to $n = 2$ as

$$n = 2 + \Delta n_{th}, \ | \Delta n_{th} | \ll 2. \tag{88}$$

Expanding

$$f_n \propto n^{-3} \tag{89}$$

about $n = 2$ yields

$$\frac{\Delta f}{f_2} \simeq -\frac{3}{2}\Delta n_{th}. \tag{90}$$

Hence

$$\Delta n_{th} \simeq -\frac{2}{3}\frac{\Delta f}{f_2}. \tag{91}$$

Here $\Delta n_{th}$ is a perturbative representation of a theoretically calculated energy or frequency shift. It should not be confused with a continuously varying fundamental quantum number or with $n_{obs}$ derived from the measured waveform.

# 6 Gravitational-Wave Emission from Quantized NP States

In this section we address the question on how a system with a discrete gravitational eigenbasis can be reconciled with the continuously evolving gravitational-wave chirp observed from astrophysical binary black holes. This question has two qualitatively different regimes. That is the large-$n$ semiclassical inspiral and the low-$n$ strongly quantum regime approaching the proposed $n = 2$ terminal state.

## 6.1 Large-$n$ correspondence and the early inspiral

At large separation, the binary occupies gravitational configurations characterized by large values of $n$ such that $r_n = n^2 r_g$ The separation between neighbouring eigen-radii is therefore

$$\Delta r_n = r_{n+1} - r_n = (2n+1)r_g. \tag{92}$$

Relative to $r_n$,

$$\frac{\Delta r_n}{r_n} = \frac{2n+1}{n^2}, \tag{93}$$

so that

$$\frac{\Delta r_n}{r_n} \simeq \frac{2}{n} \to 0, \ n \to \infty. \tag{94}$$

Thus, although the underlying spectrum remains discrete, the relative spacing of neighbouring geometrical states becomes progressively smaller at large $n$.

The corresponding gravitational-wave frequency is

$$f_n = \frac{c^3}{\pi G M n^3}. \tag{95}$$

For neighboring high-$n$ states,

$$\frac{f_{n-1}}{f_n} = \left(\frac{n}{n-1}\right)^3, \tag{96}$$

which approaches unity as $n \to \infty$. Expanding for large $n$,

$$\left(\frac{n}{n-1}\right)^3 = 1 + \frac{3}{n} + \mathscr{O}\left(n^{-2}\right). \tag{97}$$

Hence the fractional frequency separation between neighboring levels becomes small,

$$\frac{\Delta f_n}{f_n} \sim \frac{3}{n},\ n \gg 1. \tag{98}$$

The large-$n$ regime therefore provides a natural correspondence limit in which the discrete NP spectrum becomes increasingly dense relative to the characteristic orbital scale. In this limit, a sequence of closely spaced quantum-geometrical states may approximate the continuous evolution familiar from classical binary dynamics.

Schematically, $n \to \infty \implies \frac{\Delta r_n}{r_n} \to 0,\ \frac{\Delta f_n}{f_n} \to 0$

This correspondence limit provides a possible connection between the discrete NP description and the approximately continuous gravitational-wave chirp during the early inspiral.

It does not, however, resolve the behaviour of the system near merger, where $n$ is small.

## 6.2 Low-$n$ discreteness and the late inspiral

The large-$n$ approximation breaks down as the binary approaches the lowest members of the spectrum. This is particularly important because the terminal-state hypothesis developed in Section 5 concerns $n = 2$ At such small values of $n$, neighboring levels are widely separated.

For example, Equation (98) gives

$$\frac{f_3}{f_4} = \left(\frac{4}{3}\right)^3 = \frac{64}{27} \simeq 2.37, \tag{99}$$

while

$$\frac{f_2}{f_3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8} = 3.375. \tag{100}$$

Consequently, a literal interpretation in which the late binary occupies a single stationary eigenstate and undergoes instantaneous transitions

$$4 \to 3 \to 2 \tag{101}$$

would imply large discrete changes in the characteristic frequencies.

Such isolated jumps are not the phenomenology of the observed late-inspiral and merger waveform. The large-$n$ correspondence argument therefore cannot simply be extrapolated into the $n \simeq 2$ regime.

This is precisely where the discrete character of the NP spectrum becomes observationally consequential.

The late-time gravitational state must therefore be treated dynamically rather than as a sequence of infinitely sharp stationary levels.

## 6.3 Time-dependent superposition of gravitational eigenstates

A discrete eigenvalue spectrum does not require a dynamical system to occupy a single eigenstate at every instant. The time-dependent gravitational configuration may instead be expanded in the NP eigenbasis as

$$|\Psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} \mid n \Big\rangle, \tag{102}$$

where $c_n(t)$ are time-dependent amplitudes satisfying the normalization condition

$$\sum_n \mid c_n(t) \mid^2 = 1. \tag{103}$$

The discreteness remains in the basis, $n \in \mathbb{N}$, while the physical state evolves continuously through the amplitudes $c_n(t)$.Thus, the evolving binary is represented as a time-dependent superposition of discrete NP states.

The expectation value of the mass-quadrupole operator is

$$Q_{ij}(t) = \left\langle \Psi(t) \mid \widehat{Q}_{ij} \mid \Psi(t) \right\rangle. \tag{104}$$

Substitution of Equation (102) gives

$$Q_{ij}(t) = \sum_{m,n} c_m^*(t) c_n(t) Q_{ij}^{mn} e^{-i(E_n - E_m)t/\hbar}, \tag{105}$$

where

$$Q_{ij}^{mn} = \left\langle m \mid \widehat{Q}_{ij} \mid n \right\rangle. \tag{106}$$

The gravitational-wave signal is therefore associated with the collective time dependence of the quadrupole expectation value rather than with a single continuously varying quantum number.

The characteristic transition frequencies are

$$\omega_{mn} = \frac{E_n - E_m}{\hbar}. \tag{107}$$

Accordingly, a time-dependent superposition may contain several simultaneously contributing transition frequencies, with amplitudes determined by $c_m^* c_n Q_{ij}^{mn}$.

This provides a possible quantum mechanism by which an underlying discrete spectrum could generate a more complicated radiative waveform.

## 6.4 Quadrupolar radiative transitions

The leading radiative channel of a compact gravitational system is quadrupolar. A transition between NP gravitational states may therefore be represented schematically as

$$|\Psi_i\rangle \longrightarrow |\Psi_f\rangle + h_{GW}, \tag{108}$$

with transition energy

$$\Delta E_{fi} = E_i - E_f. \tag{109}$$

For a single emitted gravitational quantum,

$$\hbar\omega_{fi} = E_i - E_f. \tag{110}$$

The radiative transition amplitude is determined by the interaction Hamiltonian,

$$\mathscr{M}_{fi} = \left\langle \Psi_f \mid \widehat{H}_{rad} \mid \Psi_i \right\rangle. \tag{111}$$

At leading quadrupolar order this contains the matrix element

$$Q_{ij}^{fi} = \left\langle \Psi_f \mid \widehat{Q}_{ij} \mid \Psi_i \right\rangle. \tag{112}$$

The transition rate may be written generically in Fermi-golden-rule form,

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar} \left| \left\langle \Psi_f \mid \widehat{H}_{rad} \mid \Psi_i \right\rangle \right|^2 \rho\left(\omega_{fi}\right), \tag{113}$$

where $\rho\left(\omega_{fi}\right)$ denotes the density of accessible radiative states.

Equation (113) is presented as the general quantum-transition structure. For a rank-two quadrupole operator, the orbital-angular-momentum selection rule is

$$\Delta l = 0, \pm 2, \tag{114}$$

subject to the appropriate angular-momentum and parity restrictions.

### 6.5 Transition widths and waveform smoothness

A state with a finite lifetime $\tau_n$ possesses a characteristic linewidth of order

$$\Gamma_n \sim \frac{1}{\tau_n}, \tag{115}$$

when $\Gamma_n$ is expressed as an angular-frequency width. Equivalently, if an energy width is used,

$$\Delta E_n \sim \frac{\hbar}{\tau_n}. \tag{116}$$

Let the spacing between neighboring transition frequencies be

$$\Delta\omega_n = |\omega_{n,n-1} - \omega_{n-1,n-2}|. \tag{117}$$

Two qualitatively different regimes then arise.

If $\Gamma_n \gtrsim \Delta\omega_n$ neighbouring transitions overlap substantially and may not appear as individually resolved spectral features. Conversely, if $\Gamma_n \ll \Delta\omega_n$ the underlying discrete transitions should, in principle, produce distinguishable spectral or temporal structure. Equations (116) and (117) therefore provide a direct consistency test of the model.

## 7 Relativistic and Spin Corrections

While the leading-order Nexus Paradigm predicts a clean $n = 2$ merger state with $v = c/2$, astrophysical binaries experience significant relativistic and spin effects near merger. We incorporate these as perturbations to the effective Hamiltonian.

### 7.1 Relativistic Kinetic Energy Correction

The relativistic kinetic energy of the binary system with reduced mass $\mu$ is

$$K = \sqrt{(pc)^2 + (\mu c^2)^2} - \mu c^2 \approx \frac{p^2}{2\mu} - \frac{p^4}{8\mu^3 c^2} + \cdots. \tag{118}$$

The second term defines the correction operator

$$H_1 = -\frac{p^4}{8\mu^3 c^2}. \tag{119}$$

The term in Eq. (119) represents the leading special-relativistic correction to the non-relativistic kinetic-energy expansion. Its contribution to the terminal orbital frequency cannot be inferred from the sign of the Hamiltonian correction alone; the corresponding shift must instead be obtained from the corrected orbital dynamics. We therefore denote its contribution to the NP terminal-frequency relation by $\delta_{rel}$, to be calculated independently of the observed peak frequency

### 7.2 Spin–Orbit Coupling and the Perturbation of the $n = 2$ Terminal State

For a binary with component masses $m_1$ and $m_2$, the individual black-hole spins are written in terms of the dimensionless Kerr parameters $\chi_i$ as

$$S_i = \chi_i \frac{G m_i^2}{c} \widehat{S}_i, \ | \ \chi_i \ | \leq 1. \tag{120}$$

This distinction is essential in the sense that $\chi_i$ is dimensionless, whereas $S_i$ carries the dimensions of angular momentum.

At leading post-Newtonian order, the conservative spin-orbit interaction of a compact binary has the generic Hamiltonian structure

$$H_{SO} = \frac{G}{c^2 R^3} L \cdot [C_1(q) S_1 + C_2(q) S_2], \tag{121}$$

where $R$ is the binary separation, $L$ is the orbital angular momentum, $q = m_2/m_1 \leq 1$, and $C_1(q)$ and $C_2(q)$ are mass-ratio-dependent coefficients. Standard PN and EOB treatments possess precisely this $L \cdot S$ structure and describe spin-orbit effects as perturbations of the two-body Hamiltonian.

For an NP binary eigenstate $\Psi_{nlm}$, the leading spin-orbit energy correction is consequently

$$\Delta E_{SO} = \langle \Psi_{nlm} | H_{SO} | \Psi_{nlm} \rangle . \tag{122}$$

The sign of this correction is not universal. It depends on the projections of the two spins onto the orbital angular momentum. Aligned and anti-aligned configurations therefore cannot consistently be represented by a correction that always increases the terminal velocity or merger frequency. This is particularly important because the observed BBH population contains both positive and negative values of the effective spin parameter.

For comparison with observations, the commonly measured effective aligned spin is

$$\chi_{eff} = \frac{m_1 \chi_1 \cos\theta_1 + m_2 \chi_2 \cos\theta_2}{M}, \; M = m_1 + m_2, \tag{123}$$

where $\theta_i$ denotes the angle between $S_i$ and $L$. Thus $\chi_{eff} > 0$ generally corresponds to a net aligned spin contribution and $\chi_{eff} < 0$ to a net anti-aligned contribution. More complete treatments require both component spins because a single $\chi_{eff}$ does not uniquely determine the full spin-orbit Hamiltonian. Contemporary EOB descriptions accordingly retain explicit spin-dependent contributions to the binary Hamiltonian.

**Perturbation of the NP frequency**

A first-order expansion about $n = 2$ gives

$$\frac{\Delta f}{f_2} \simeq -\frac{3}{2}\Delta n. \tag{124}$$

The total theoretical perturbation may therefore be decomposed schematically as

$$\Delta n_{th} = \Delta n_{rel} + \Delta n_q + \Delta n_{SO} + \Delta n_e + \cdots, \tag{125}$$

where the terms represent, respectively, relativistic, finite-mass-ratio, spin-orbit and eccentricity corrections. Equivalently, $f_{peak}^{NP} = f_2^{NP}(1 + \delta_{rel} + \delta_q + \delta_{SO} + \delta_e + \cdots)$ with

$$\delta_i \simeq -\frac{3}{2}\Delta n_i. \tag{126}$$

Equation (126) shows an important separation between prediction and observation. Every $\delta_i$ must be calculated from independently measured or theoretically specified binary parameters before comparison with $f_{peak}^{obs}$. The observed peak frequency is therefore not used to determine the theoretical correction.

For diagnostic purposes only, the measured frequency can subsequently be represented by $n_{obs} = \left(\frac{c^3}{\pi G M f_{peak}^{obs}}\right)^{1/3}$ or by the fractional residual $\delta_f^{obs} = \frac{f_{peak}^{obs} - f_2^{NP}}{f_2^{NP}}$.

Neither $n_{obs}$ nor $\delta_f^{obs}$ is an adjustable NP parameter. They quantify how far an event lies from the zeroth-order $n = 2$ prediction.

This distinction is shown in Table 2. For example, GW190412 lies very close to the unperturbed $n = 2$ frequency, whereas several other BBH events exhibit positive frequency residuals. Because these residuals occur for systems with both positive and negative $\chi_{eff}$, they cannot in general be attributed to spin-orbit coupling alone. The appropriate interpretation is therefore that spin-orbit coupling supplies one signed contribution, while relativistic strong-field, finite-mass-ratio and possibly eccentricity corrections provide additional contributions. Standard relativistic binary dynamics likewise contains several distinct conservative corrections rather than a single spin term; spin-orbit coupling first enters as a specific higher-order contribution to the Hamiltonian.

## 7.3 Direct comparison with gravitational-wave observations

Table 2 tests the zeroth-order $n = 2$ prediction without defining an effective quantum number from the observed velocity. For each event, the source-frame total mass $M$, mass ratio $q$, and effective spin $\chi_{eff}$ are independently determined from the LVK parameter-estimation analyses. The unperturbed NP prediction is then calculated solely from $f_2^{NP} = \frac{c^3}{8\pi GM}$. The fractional frequency residual is defined by $\delta_f \equiv \frac{f_{peak}^{obs} - f_2^{NP}}{f_2^{NP}}$, and, for diagnostic purposes

Table 2: LIGO/Virgo/KAGRA Black Hole Merger Inspiral

| **Event** | $M\,[M_\odot]$ | $q = m_2/m_1$ | $\chi_{eff}$ | $f^{obs}_{peak}$**(Hz)** | $f_2^{NP}$**(Hz)** | $\Delta f/f_2$ | $n_{obs}$ |
|---|---|---|---|---|---|---|---|
| GW150914 | $64.5^{+3.7}_{-3.2}$ | 0.87 | $-0.04^{+0.12}_{-0.14}$ | 150 | 125.2 | +0.198 | 1.883 |
| GW151226 | $\sim 21.4$ | 0.56 | $+0.18^{+0.20}_{-0.12}$ | 450 | 377.5 | +0.192 | 1.886 |
| GW170104 | $49.6^{+4.7}_{-3.6}$ | $\sim 0.65$ | $-0.04^{+0.15}_{-0.19}$ | 200 | 162.8 | +0.229 | 1.867 |
| GW170814 | $\sim 55-56$ | $\sim 0.82$ | $+0.08^{+0.13}_{-0.12}$ | 200 | $\sim 145$ | $\sim +0.38$ | $\sim 1.80$ |
| GW190412 | $38.4^{+3.8}_{-3.7}$ | 0.28 | $+0.25^{+0.08}_{-0.11}$ | 211 | 210.4 | +0.003 | 1.998 |
| GW190521 | $153.1^{+42.2}_{-16.2}$ | 0.58 | $-0.14^{+0.50}_{-0.45}$ | 60 | 52.8 | +0.137 | 1.916 |

*Note:*

- Masses in solar masses ($M_\odot$)
- $f^{obs}_{peak} \equiv$ instantaneous GW frequency at the maximum of $|h_{22}(t)|$

only, the observed frequency may be mapped onto $n_{obs} = 2\left(1+\delta_f\right)^{-1/3} = \left(\frac{c^3}{\pi G M f^{obs}_{peak}}\right)^{1/3}$. Importantly, $n_{obs}$ is not fitted back into the NP frequency relation and is not regarded as a theoretical prediction. It merely quantifies the departure of each event from the unperturbed $n = 2$ terminal state.

# 8 Conclusions

We have developed a quantum-geometrical description of binary black-hole coalescence within the Nexus Paradigm (NP), in which compact objects are represented by quantized Ricci-soliton states with $r_n = n^2 r_g,\ v_n = \frac{c}{n}$. The binary is formulated as a two-centre gravitational eigenvalue problem whose collective states evolve along an adiabatic energy surface. The merger separation $R_c$ is therefore a dynamical quantity and should not be identified directly with the eigen-radius $r_n$.

Our current model proposes $n = 2$ as the lowest admissible massive terminal state, giving $r_2 = \frac{4GM}{c^2},\ v_2 = \frac{c}{2}$, and the principal zeroth-order prediction $f_2^{NP} = \frac{c^3}{8\pi GM}$ or equivalently $M f_2^{NP} = \frac{c^3}{8\pi G}$.

This relation provides a direct test using independently measured source masses and gravitational-wave peak frequencies, without introducing an observationally fitted $n_{eff}$. Deviations from the zeroth-order relation should instead be tested against independently derived relativistic, mass-ratio, spin-orbit and eccentricity corrections.

The discreteness of the NP spectrum also provides an important observational test. While the $n \gg 1$ regime approaches a semiclassical continuous limit, the low-$n$ spectrum near merger is strongly discrete. A smooth late-time waveform therefore requires sufficiently overlapping quantum transitions, approximately satisfying $\Gamma_n \gtrsim |\omega_{n,n-1} - \omega_{n-1,n-2}|$. Calculating these transition widths, the two-soliton interaction, and the higher-order corrections remains an important direction for future work.

Thus, the principal falsifiable predictions of the model are the $n = 2$ mass-frequency relation and the possible signatures of low-$n$ spectral discreteness in gravitational-wave signals. These provide direct avenues for testing the proposed quantum structure of black-hole coalescence with present and future gravitational-wave observations.

**Data Availability**

The gravitational-wave strain data and publicly released parameter-estimation products used in this study are available through the Gravitational Wave Open Science Center (GWOSC) and the corresponding LIGO–Virgo–KAGRA catalogue releases cited in the text. Derived quantities reported in this work are calculated from these publicly available data.

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# Conflict of interest

We declare no conflict of interest.

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