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CHAPTER 8

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Alnadhief H. A. Alfedeel

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Keywords: Scale-dependent gravity; running couplings; Bianchi I; anisotropic cosmology

Anisotropic Model in Scale-dependent Gravity

Alnadhief H. A. Alfedeel[1,2]

[1]Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 13318, Saudi Arabia

[2]Centre for Space Research, North-West University, Potchefstroom 2520, South Africa

E-mail: aaalnadhief@imamu.edu.sa

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We investigate the cosmological dynamics of a scale-dependent (SD) gravitational framework in which both the Newton coupling G_k and the cosmological term Λ_k evolve with cosmic time. The analysis is carried out in a spatially homogeneous but anisotropic Bianchi I Universe, allowing us to probe the interplay between anisotropic expansion and running gravitational couplings. By employing the dimensionless variables $h = H/H_0$, $g = G/G_0$, and the density parameters Ω_m, Ω_r, and Ω_σ, the modified field equations are recast into a closed system of first-order differential equations in redshift space and numerically integrated using a fourth-order Runge–Kutta method with initial conditions qualitatively consistent with current cosmological constraints. The obtained solutions reproduce the expected early-time behavior dominated by matter and radiation while yielding a late-time accelerating expansion driven by the dynamical running of G_k and Λ_k. The effective equation of state approaches a quintessence-like regime with $w_{\rm eff}(0) \simeq -0.70$, and the present deceleration parameter $q_0 \approx -0.60$ lies within the range inferred from observational data. The negative slope of the $Om(z)$ diagnostic further confirms that the model behaves as a quintessence-like dark-energy scenario rather than a cosmological-constant model. Altogether, the results demonstrate that scale-dependent gravitational couplings in an anisotropic background not only reproduce the essential phenomenology of ΛCDM at low redshift, but also introduce controlled and observationally admissible deviations that provide a compelling alternative description of late-time cosmic acceleration.

1 Introduction

The discovery of the late-time accelerated expansion of the Universe, first revealed by Type Ia supernova observations [1, 2] and later supported by measurements of the cosmic microwave background (CMB) anisotropies [3] and baryon acoustic oscillations (BAO) [4], has become one of the central challenges in modern cosmology. Within the standard cosmological model, the acceleration of the Universe is attributed to a cosmological constant Λ, giving rise to the successful ΛCDM paradigm. Despite its empirical success, ΛCDM suffers from severe theoretical shortcomings, most notably the fine-tuning problem, arising from the enormous discrepancy between the observed value of Λ and estimates of vacuum energy from quantum field theory–and the coincidence problem, which questions why the dark-energy component begins to dominate the energy budget only in the current cosmological epoch [5–7]. These theoretical conflicts encourage us to look into modified-gravity models and dynamical dark-energy scenarios as possible replacements for the cosmological constant. Quantum gravity (QG) and effective field theory (EFT) offer a theoretically interesting alternative, wherein the gravitational running coupling, namely Newton's coupling G and the cosmological constant Λ, acquire a nontrivial dependence on the renormalization-group (RG) scale k [8–11]. The gravitational couplings change as the Universe expands, and this scale is inherently linked to the Hubble rate of expansion $k \propto H(z)$, in cosmological contexts. The resulting framework—often referred to as scale-dependent (SD) gravity—modifies Einstein's equations by introducing additional contributions proportional to derivatives of the running couplings. These contributions can be interpreted as an effective dark-energy component that arises directly from the quantum or scale-dependent structure of gravity, without the need to introduce *ad hoc* scalar fields or exotic matter sources [12]. The cosmological implications within the framework of scale-dependent (SD) gravity have been extensively examined by several authors. For example, a novel covariant formulation of the SD setting technique was developed in the analysis of the field equations in [13]. Two models for the evolution of the gravitational and cosmological couplings were developed from the renormalization group-equations while enforcing the conservation of the energy–momentum tensor. One paradigm circumvents the Big Bang singularity, whereas the other mitigates the Hubble tension. Furthermore, fundamental solutions for structure formation were developed and qualitatively analyzed at the perturbative level. Reference [14] presented a

cosmological model based on SD. The authors argued that the models have the potential to address the tensions between early and late time measurements of the Hubble parameter H_0. Analysis of tension characterization criteria reveals that the tension can be alleviated. Moreover, the phenomenological implications of the recently suggested concept of a running gravitational coupling on macroscopic scales have been investigated by [15]. In accordance with that, [16] has examined the observational implications of an SD model, in which both G_k and Λ_k only run at the perturbation level.

The model's free parameters are constrained using the CMB. SNe Ia, BAO, and RSD data. It is found that the GR correction is consistent with ΛCDM predictions. However, it does not significantly improve the argument between early- and lat-time cosmological observations. Most investigations of SD gravity have been performed within isotropic and homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) backgrounds, with a limited number extending beyond the standard model or involving non-perfect fluid distributions. However, the early Universe may have possessed small but non-negligible anisotropies, as suggested by certain large-scale CMB anomalies and possible hints of Bianchi patterns [17, 18]. Moreover, anisotropic geometries provide a more general arena in which to test the stability and robustness of modified-gravity models. The Bianchi type$-I$ spacetime provides a natural setting for investigating the persistence of SD effects when the assumption of isotropy is relaxed, since it is the simplest anisotropic generalization of the spatially flat FLRW metric.

In this paper, we consider SD gravity as a framework for studying the Bianchi type$-I$ cosmological model. We derive the modified field equations (MFEs) in terms of dimensionless variables and integrate the resulting dynamical system using observationally motivated initial conditions. To diagnose the cosmological behavior of the model, we employ two well-established, model-independent tools: the state-finder parameters (r, s) [19] and the $Om(z)$ diagnostic [20]. These diagnostics are particularly useful for distinguishing dynamical dark-energy models from ΛCDM, since ΛCDM corresponds to the fixed point $(r, s) = (1, 0)$ and predicts a constant $Om(z) = \Omega_{m0}$. We then compare the predictions of our model with the above benchmarks to determine if the running Newton coupling can reproduce late-time acceleration, mimic quintessence-like behavior, and be consistent with current cosmological constraints.

This manuscript is structured as follows: Sec. 2 introduces the theoretical framework of the SD gravity. Sec. 3 is devoted to the formulation of the Bianchi type$-I$ cosmological model, where the anisotropic metric is introduced and the corresponding MFEs are derived. In Sec. 4, the cosmological equations are reformulated in terms of dimensionless variables and transformed into redshift space, yielding a closed autonomous system suitable for numerical integration. The numerical solutions and main results are presented in Sec. 5, which also provides a detailed discussion about the dynamical behavior of the model including the evolution of anisotropy, density parameters, and cosmographic quantities. In Sec. 6, the cosmological implications are further explored through the state-finder parameters and $Om(z)$ diagnostics to distinguish the model from the standard ΛCDM scenario. Finally, Sec. 7 summarizes the main findings of the study and outlines possible directions for further research within the scale-dependent gravity paradigm.

2 Scale-dependent gravity framework

In the scale-dependent gravity (SD) scenario, the gravitational couplings of the Einstein–Hilbert action are no longer treated as fundamental constants, but as running quantities motivated by quantum field theory and renormalizatic group (RG) methods. In particular, the Newton coupling G_0 and the cosmological term Λ_0 acquire a nontrivial dependence on a renormalization scale k, and are promoted to functions G_k and Λ_k. This scale dependence is expected on general grounds in quantum gravity and effective field theory, where all couplings "run" with the energy (or length) scale at which the theory is probed [21–25].

A convenient way to encode this running in a cosmological setting is to consider a SD version of the Einstein–Hilbert action of the form

$$S[g_{ab}, k] = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G_k} \left(R - 2\Lambda_k \right) + \mathcal{L}_m \right] , \tag{1}$$

where G_k is the SD gravitational coupling, Λ_k is the running cosmological term, R is the Ricci scalar, and $\mathcal{L}_m$ is the matter Lagrangian density. Here k is an effective renormalization scale, which is subsequently related to cosmological quantities through an appropriate scale-setting prescription. In this paper, we adopt the scale identification $k \propto H$, which is one of the most widely used choices in RG-improved cosmology, because the Hubble parameter is the typical infrared energy scale driving the cosmic background evolution. This identification preserves the spatial homogeneity of the cosmological background and has been frequently adopted in asymptotically

safe and scale-dependent gravity. Alternative identifications, e.g. $k \propto \sqrt{R}$, $k \propto 1/t$, or scales based on different curvature invariants, have also been offered in the literature. The RG scale is not fixed by basic principles and alternative choices might produce quantitative changes in the cosmic history, while the qualitative behavior is generally preserved. A systematic comparison of different scale-setting prescriptions is beyond the scope of the present work and is left for future investigation [26–29].

Varying this action with respect to the metric g_{ab} at fixed scale k leads to a modified set of Einstein field equations,

$$G_{ab} + \Lambda_k\, g_{ab} = 8\pi G_k\, T^{\rm eff}_{ab}\ , \tag{2}$$

where

$$8\pi\, T^{\rm eff}_{ab} = 8\pi G_k\, T_{ab} - \Delta t_{ab} \tag{3}$$

defines an effective energy–momentum tensor which includes, in addition to the ordinary matter content T_{ab}, a purely geometric contribution Δt_{ab} due to the running of G_k. This extra symmetric rank-two tensor is given by

$$\Delta t_{ab} = G_k\, (g_{ab}\Box - \nabla_a\nabla_b)\, G_k^{-1}\ , \tag{4}$$

where ∇_a is the covariant derivative compatible with g_{ab} and $\Box = g^{ab}\nabla_a\nabla_b$ is the d'Alembert operator. The tensor Δt_{ab} vanishes when G_k is constant and therefore encodes the dynamical effects of scale dependence at the level of the field equations. In cosmology, this contribution can be interpreted as an effective, generally anisotropic dark-energy sector generated by the running of the gravitational coupling.

Taking the covariant divergence of Eq. (2) and using the contracted Bianchi identity $\nabla^a G_{ab} = 0$ yields

$$\nabla^a \left(8\pi G_k T^{\rm eff}_{ab} - \Lambda_k g_{ab}\right) = 0\ . \tag{5}$$

If the ordinary matter sector is assumed to be separately conserved, $\nabla^a T_{ab} = 0$, then the above relation translates into a constraint that couples the running of G_k and Λ_k to the geometry via Δt_{ab}. In other words, the SD sector behaves as an effective fluid whose dynamics is determined by the RG running and by the background spacetime.

A particularly useful additional constraint is obtained by imposing a *null condition* on the SD contribution Δt_{ab}, namely

$$\Delta t_{ab}\, \ell^a \ell^b = 0 \tag{6}$$

for any null vector ℓ^a satisfying $g_{ab}\ell^a\ell^b = 0$. Physically, this condition requires that the effective stress–energy arising from the running gravitational coupling G_k does not generate additional focusing or defocusing of null geodesics beyond that produced by the standard matter sector. Specifically, imposing $\Delta_{ab}\ell^a\ell^b = 0$, where ℓ^a is an arbitrary null vector, ensures that the SD correction preserves the standard null focusing behavior of General Relativity (GR) whenever the matter fields satisfy the null energy condition. Consequently, the propagation of null congruences remains determined solely by the physical matter content, preventing the running of G_k from introducing nonphysical geometric effects. Furthermore, this condition guarantees a smooth recovery of the GR limit when the scale-dependent corrections become minimal, thereby providing a physically principled and mathematically consistent closure condition that has been widely accepted in the SD gravity and related modified gravity theories [30–33]. Using the explicit form of Δt_{ab}, Eq. (6) can be written as

$$\Delta t_{ab}\, \ell^a \ell^b = -\, G_k\, \ell^a\ell^b \nabla_a\nabla_b G_k^{-1} = 0\ , \tag{7}$$

which constrains the allowed functional form of G_k along the null directions.

In a spatially homogeneous spacetime such as the Bianchi type-I Universe considered in Sec. 3, where G_k depends only on the cosmic time t, the null condition (6) reduces to a second-order differential equation for $G_k(t)$. For the metric (9) one finds

$$\frac{\ddot{G}_k}{G_k} - 2\frac{\dot{G}_k^2}{G_k^2} - \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\frac{\dot{G}_k}{G_k} = 0\ , \tag{8}$$

which coincides with Eq. (23) obtained from the Bianchi identity. This equation plays the role of an evolution (or consistency) equation for the running Newton coupling in the cosmological background and will be used, together with the modified Friedmann and Raychaudhuri equations, to close the system and determine the dynamics of the SD Bianchi type-I model.

3 Bianchi Type-I Model

We consider a spatially homogeneous but anisotropic Universe described by the Bianchi type-I line element

$$ds^2 = -dt^2 + A^2(t)\,dx^2 + B^2(t)\,dy^2 + C^2(t)\,dz^2 \,, \tag{9}$$

where $A(t)$, $B(t)$ and $C(t)$ are the directional scale factors along the x-, y- and z-axes, respectively. This geometry generalizes the flat FLRW metric by allowing directional dependence in the expansion, while preserving spatial homogeneity. It is therefore the simplest laboratory to study how anisotropic expansion and SD couplings affect cosmic dynamics.

We assume that spacetime is filled with a perfect fluid whose energy–momentum tensor is

$$T_{ab} = (\rho + p)\,u_a u_b + p\,g_{ab} \,,$$

where g_{ab} is the metric tensor, $u^a = (1,0,0,0)$ is the fluid four-velocity in comoving coordinates satisfying $u^a u_a = -1$, ρ is the total energy density, and p is the isotropic pressure. The matter sector is further characterized by a barotropic equation of state (EoS)

$$p = w\,\rho \,, \tag{10}$$

where $w \in [-1,1]$ is the (effective) EoS parameter. Radiation is characterized by $w_r = 1/3$, pressureless matter by $w_m = 0$, and a cosmological-constant-like component by $w = -1$. In an anisotropic geometry, it is convenient to introduce several kinematical quantities that characterize the expansion:

- The average scale factor $a(t)$,
$$a(t) = (ABC)^{1/3} \,,$$
which encodes the overall volume expansion.

- The mean Hubble parameter
$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \,,$$
and the associated deceleration parameter
$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} \,, \tag{11}$$
which measures whether the Universe undergoes accelerated ($q < 0$) or decelerated ($q > 0$) expansion.

- The expansion scalar
$$\theta = \nabla_i u^i = 3H \,, \tag{12}$$
which quantifies the rate of change of the comoving volume element, and is related to the volume $V \propto ABC$ through $\dot{V}/V = \theta$.

- The average anisotropy parameter
$$A_p = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \,, \tag{13}$$
which vanishes in the isotropic limit $H_1 = H_2 = H_3 = H$ and provides a dimensionless measure of the deviation from FLRW.

- The shear scalar, which characterizes the distortion of the expansion,
$$\sigma = \frac{\sigma_0}{a^3} \,, \tag{14}$$
where σ_0 is a constant. The scaling $\sigma \propto a^{-3}$ implies that anisotropies decay rapidly as the Universe expands, so a Bianchi I Universe tends to isotropize at late times, in agreement with the observed near-isotropy of the CMB.

In terms of the directional Hubble rates $H_1 = \dot{A}/A$, $H_2 = \dot{B}/B$ and $H_3 = \dot{C}/C$, we have

$$3H = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \,, \tag{15}$$

as expected.

The Einstein field equations (2) for the metric (9) in SD gravity can be decomposed into four independent equations:

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} = 8\pi G_k \rho + \Lambda_k - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\frac{\dot{G}_k}{G_k} \,, \tag{16}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -8\pi G_k p + \Lambda_k + \left(\frac{\ddot{G}_k}{G_k} - 2\frac{\dot{G}_k^2}{G_k^2}\right) + \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\frac{\dot{G}_k}{G_k} \,, \tag{17}$$

$$\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} = -8\pi G_k p + \Lambda_k + \left(\frac{\ddot{G}_k}{G_k} - 2\frac{\dot{G}_k^2}{G_k^2}\right) + \left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\frac{\dot{G}_k}{G_k} \,, \tag{18}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -8\pi G_k p + \Lambda_k + \left(\frac{\ddot{G}_k}{G_k} - 2\frac{\dot{G}_k^2}{G_k^2}\right) + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\frac{\dot{G}_k}{G_k} \,, \tag{19}$$

where G_k and Λ_k denote the running Newton coupling and the cosmological term, respectively. The extra terms proportional to the derivatives of G_k encode the backreaction of the RG running at the level of the field equations and effectively behave as an anisotropic dark-energy sector.

Taking the divergence of the Einstein tensor in Eq. (2) yields an auxiliary equation (the contracted Bianchi identity in the presence of running couplings)

$$8\pi G_k \left[\dot{\rho} + (\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right] + 8\pi\rho\dot{G}_k + \dot{\Lambda}_k - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\left[\frac{\ddot{G}_k}{G_k} - \frac{\dot{G}_k^2}{G_k^2}\right] - \left[\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) - \left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right)\right]\frac{\dot{G}_k}{G_k} = 0 \,. \tag{20}$$

Assuming that the ordinary matter energy–momentum tensor is separately conserved, $\nabla_a T^{ab} = 0$, the above equation can be split into two parts, namely a standard continuity equation for the fluid

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0 \,, \tag{21}$$

and an evolution equation that couples the running of G_k and Λ_k to the anisotropic expansion,

$$8\pi\rho\dot{G}_k + \dot{\Lambda}_k - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\left[\frac{\ddot{G}_k}{G_k} - \frac{\dot{G}_k^2}{G_k^2}\right] - \left[\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) - \left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right)\right]\frac{\dot{G}_k}{G_k} = 0 \,. \tag{22}$$

This relation shows explicitly how the time dependence of G_k and Λ_k is constrained by both the matter content and the anisotropic expansion. Moreover, the null condition (6) supplies an additional relation that can be used to close the system:

$$\frac{\ddot{G}_k}{G_k} - 2\frac{\dot{G}_k^2}{G_k^2} - \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\frac{\dot{G}_k}{G_k} = 0 \,. \tag{23}$$

Combining the G_{00} equation with suitable linear combinations of the spatial components, and rewriting in terms of the mean Hubble parameter $H = \dot{a}/a$ and shear scalar σ^2, one obtains

$$3H^2 - \sigma^2 = 8\pi G_k \rho_m + 8\pi G_k \rho_r + \Lambda_k + 3H\frac{\dot{G}_k}{G_k} \,, \tag{24}$$

$$2\dot{H} + 3H^2 + \sigma^2 = -8\pi G_k p_m - 8\pi G_k p_r + \Lambda_k + \left(\frac{\ddot{G}_k}{G_k} - 2\frac{\dot{G}_k^2}{G_k^2}\right) + 2H\,\frac{\dot{G}_k}{G_k} \,. \tag{25}$$

To simplify the interpretation of the SD contributions, we combine the cosmological term, the running of Newton's coupling, and the anisotropic shear into an effective running vacuum component defined by

$$\lambda_k \equiv \Lambda_k + 3H\frac{\dot{G}_k}{G_k} + \sigma^2 \, . \tag{26}$$

The corresponding effective energy density and pressure are

$$\rho_\lambda = \frac{\lambda_k}{8\pi G_k} \, , \qquad p_\lambda = -\rho_\lambda \, .$$

Here we have explicitly allowed for a decomposition of the matter content into a pressureless component (ρ_m, p_m) and a relativistic component (ρ_r, p_r), so that the model is suitable for describing the entire thermal history of the Universe. By combining Eqs. (24) and (25), we obtain the evolution equation for the Hubble parameter,

$$\dot{H} = -\frac{8\pi G_k}{2}(\rho_m + \rho_r + p_m + p_r) - \sigma^2 + \left(\frac{\ddot{G}_k}{G_k} - 2\frac{\dot{G}_k^2}{G_k^2}\right) - H\,\frac{\dot{G}_k}{G_k} \, , \tag{27}$$

which makes explicit the competing roles of matter, radiation, shear and running couplings in driving cosmic acceleration or deceleration. Using the null condition (23), this simplifies further to

$$\dot{H} = -\frac{8\pi G_k}{2}(\rho_m + p_m) - \frac{8\pi G_k}{2}(\rho_r + p_r) - \sigma^2 \, , \tag{28}$$

Equation (28) governs the evolution of the mean expansion through the average scale factor and shows that, within the effective Raychaudhuri equation, the running of the gravitational and cosmological couplings, G_k and Λ_k, is absorbed into the effective energy densities and pressures. Consequently, the background dynamics can be described in terms of these effective quantities without explicitly tracking the scale dependence in the evolution equation itself.

For completeness, we also reconstruct the directional metric functions $A(t)$, $B(t)$, and $C(t)$, which characterize the anisotropic expansion along the three spatial directions. Although these metric functions are not directly observable, they determine the directional Hubble expansion rates and the shear scalar, thereby providing the geometrical description of anisotropy and its evolution toward the isotropic FLRW limit. Since the primary focus of the present work is the mean cosmological evolution and the isotropization process, the detailed derivation of the directional solutions is presented in Appendix A.

To obtain the explicit forms of the directional scale factors, we subtract the spatial components of the modified field equations from one another, thereby eliminating the common isotropic contribution and yielding a set of first-order differential equations governing the anisotropic evolution.

$$\frac{\dot{C}}{C} - \frac{\dot{A}}{A} = \frac{k_1 G_k}{ABC} \, , \tag{29}$$

$$\frac{\dot{C}}{C} - \frac{\dot{B}}{B} = \frac{k_2 G_k}{ABC} \, , \tag{30}$$

$$\frac{\dot{B}}{B} - \frac{\dot{A}}{A} = \frac{k_3 G_k}{ABC} \, , \tag{31}$$

where k_1, k_2 and k_3 are integration constants related to the initial anisotropy. Integrating Eqs. (29)–(31) with respect to cosmic time t and using $ABC = a^3$ yields

$$\frac{C}{A} = m_1 \exp\left\{\int \frac{k_1 G_k}{a^3}\, dt\right\} \, , \tag{32}$$

$$\frac{C}{B} = m_2 \exp\left\{\int \frac{k_2 G_k}{a^3}\, dt\right\} \, , \tag{33}$$

$$\frac{B}{A} = m_3 \exp\left\{\int \frac{k_3 G_k}{a^3}\, dt\right\} \, , \tag{34}$$

from which the metric functions A, B and C can be written in terms of $a(t)$ and the running coupling as

$$A = d_1\, a\, \exp\left\{-\frac{k_1+k_3}{3}\int \frac{G_k}{a^3}\,dt\right\}\,, \tag{35}$$

$$B = d_2\, a\, \exp\left\{\frac{2k_3-k_1}{3}\int \frac{G_k}{a^3}\,dt\right\}\,, \tag{36}$$

$$C = d_3\, a\, \exp\left\{\frac{2k_1-k_3}{3}\int \frac{G_k}{a^3}\,dt\right\}\,, \tag{37}$$

where

$$d_1 = 3\sqrt{\frac{m_2}{m_1^2}}\,, \quad d_2 = 3\sqrt{\frac{m_1}{m_2^2}}\,, \quad d_3 = 3\sqrt{m_1 m_2}\,. \tag{38}$$

Here m_1, m_2 and m_3 are integration constants related to the initial anisotropy, while the k_1, k_2 and k_3 encode the effect of the scale-dependent coupling G_k on each directional scale factor. The energy density parameters associated with matter, radiation, and the effective running-vacuum sector are defined as

$$\Omega_m = \frac{8\pi G_k \rho_m}{3H^2}\,, \qquad \Omega_r = \frac{8\pi G_k \rho_r}{3H^2}\,, \qquad \Omega_\lambda = \frac{8\pi G_k \rho_\lambda}{3H^2} = \frac{\lambda_k}{3H^2} = \frac{\Lambda_k}{3H^2} + \frac{\sigma^2}{3H^2} + \frac{\dot{G}_k}{HG_k}. \tag{39}$$

Consequently, the modified Friedmann constraint takes the normalized form

$$\Omega_m + \Omega_r + \Omega_\lambda = 1.$$

The quantity Ω_λ therefore represents the combined contribution of the running cosmological term, the shear scalar, and the variation of the gravitational coupling.

Here, λ_k is the effective contribution of the running vacuum as defined in Eq. (26), ρ_m is the energy density of matter, ρ_r denotes the energy density of radiation and σ^2 is the shear scalar. Their present values are observationally constrained, e.g., $\Omega_{m0} = 0.3166 \pm 0.0084$ and $\Omega_{\Lambda 0} = 0.6847 \pm 0.0073$ from Planck 2018 [34]. It is important to note that the Planck value $\Omega_{\Lambda 0}$ corresponds to the cosmological-constant density parameter within the standard ΛCDM model. In the present scale-dependent model, the corresponding effective quantity is $\Omega_{\lambda 0}$, which contains additional contributions from the running of G_k and from the anisotropic shear. Therefore, the Planck value may be used only as a reference value rather than as a direct observational measurement of $\Omega_{\lambda 0}$.

Differentiating Ω_m with respect to cosmic time t, and using Eq. (21), we obtain the following:

$$\dot{\Omega}_m = \left(-2\frac{\dot{H}}{H} + \frac{\dot{G}_k}{G_k}\right)\Omega_m - 3H(1+w_m)\Omega_m\,. \tag{40}$$

For the radiation density parameter Ω_r, using the conservation equation

$$\dot{\rho}_r + 3(1+w_r)H\rho_r = 0\,,$$

we find

$$\dot{\Omega}_r = \left(-2\frac{\dot{H}}{H} + \frac{\dot{G}_k}{G_k}\right)\Omega_r - 3(1+w_r)H\,\Omega_r\,. \tag{41}$$

For completeness, the evolution of the separate cosmological-term density parameter $\Omega_\Lambda = \Lambda_k/(3H^2)$ is

$$\dot{\Omega}_\Lambda = \left[1 - \Omega_m + \frac{\dot{H}}{H^2}\right]\frac{\dot{G}_k}{G_k} + \frac{1}{H}\frac{\dot{G}_k^2}{G_k^2} - 2\frac{\dot{H}}{H}\,\Omega_\Lambda\,. \tag{42}$$

Equation (42) cannot be used as the evolution equation for the newly defined effective parameter Ω_λ. Differentiating the normalized constraint

$$\Omega_m + \Omega_r + \Omega_\lambda = 1\,, \tag{43}$$

with respect to cosmic time t, gives

$$\dot{\Omega}_\lambda = -\dot{\Omega}_m - \dot{\Omega}_r\,. \tag{44}$$

Substituting Eqs. (40) and (41), we obtain

$$\dot{\Omega}_\lambda = \left(2\frac{\dot{H}}{H} - \frac{\dot{G}_k}{G_k}\right)(\Omega_m + \Omega_r) + 3H(1+w_m)\Omega_m + 3H(1+w_r)\Omega_r \ . \tag{45}$$

Using $\Omega_m + \Omega_r = 1 - \Omega_\lambda$, this can equivalently be written as

$$\dot{\Omega}_\lambda = \left(2\frac{\dot{H}}{H} - \frac{\dot{G}_k}{G_k}\right)(1 - \Omega_\lambda) + 3H(1+w_m)\Omega_m + 3H(1+w_r)\Omega_r \ . \tag{46}$$

Alternatively, the modified Friedmann constraint determines Ω_λ completely, and thus there is no need to introduce it as an independent dynamical variable. Instead, its value can be obtained algebraically at each time step from

$$\Omega_\lambda = 1 - \Omega_m - \Omega_r. \tag{47}$$

This approach automatically preserves the normalization condition throughout the evolution, avoids the accumulation of numerical errors, and provides a more stable and efficient implementation of the background dynamics.

Consequently, with the effective running vacuum definition adopted in this work, the background evolution is completely described by Eqs. (23), (28), (40), and (41), together with the algebraic constraint (47). Therefore, Eq. (42), which governs the evolution of the separate cosmological-term density parameter Ω_Λ, is no longer applicable because the effective density parameter Ω_λ incorporates not only the running cosmological term but also the contributions arising from the scale dependence of the gravitational coupling and the anisotropic shear. If one nevertheless chooses to evolve Ω_λ as an independent variable, then Eq. (42) must be replaced by either Eq. (45) or Eq. (46), which are fully consistent with the revised definition of the effective running vacuum.

4 Dimensionless Transform

The cosmological dynamics of the SD Bianchi type$-I$ model are completely specified by the background variables $\{\rho, p, \Lambda_k, G_k, H, A, B, C\}$, where A, B, and C are the directional scale factors. From these variables, the kinematical quantities σ, θ, and q are determined as defined in the previous section. In order to confront the model with observational data (e.g. SNe Ia, BAO, cosmic chronometers), it is convenient to recast the dynamical equations in terms of the redshift z, which is directly measurable.

For $a_0 = 1$, the redshift is related to the mean scale factor through

$$a(t) = \frac{1}{1+z} \ . \tag{48}$$

For any differentiable background quantity $Q(t)$ (e.g., H, G_k, Ω_m, Ω_r, or Ω_σ), the derivatives with respect to cosmic time t and redshift z are related by

$$\dot{Q} = -(1+z)HQ' \ , \qquad \ddot{Q} = (1+z)^2H^2Q'' + (1+z)^2HH'Q' + (1+z)H^2Q' \ , \tag{49}$$

where dots and primes denote derivatives with respect to t and z, respectively. We introduce the normalized Hubble parameter and Newton coupling

$$h(z) = \frac{H(z)}{H_0} \ , \qquad g(z) = \frac{G_k(z)}{G_{k0}} \ ,$$

where H_0 and G_{k0} denote their present-day values. We also define

$$Z(z) = g'(z) \ .$$

Using these expressions, the system of differential equations (23), (28), (40) and (41), together with the algebraic constraint can be written in a dimensionless form.

$$h' = \frac{h}{1+z}\left[\frac{3(1+w_m)}{2}\,\Omega_m + \frac{3(1+w_r)}{2}\,\Omega_r\right] + \frac{3h}{1+z}\,\Omega_\sigma\ , \tag{50}$$

$$\Omega_m' = \left[-2\frac{h'}{h} + \frac{Z}{g} + \frac{3(1+w_m)}{1+z}\right]\Omega_m\ , \tag{51}$$

$$\Omega_r' = \left[-2\frac{h'}{h} + \frac{Z}{g} + \frac{4}{1+z}\right]\Omega_r\ , \tag{52}$$

$$\Omega_\sigma' = \Omega_\sigma\left[\frac{6}{1+z} + 2\frac{Z}{g} - 2\frac{h'}{h}\right]\ , \tag{53}$$

$$g' = Z\ , \tag{54}$$

$$Z' = 2\frac{Z^2}{g} - \left[\frac{2}{1+z} + \frac{h'}{h}\right]Z\ , \tag{55}$$

where $w_r = 1/3$ for the radiation component. This system describes the redshift evolution of the effective background cosmology in terms of six dimensionless functions $\{h, \Omega_m, \Omega_r, \Omega_\sigma, g, Z\}$ and can be integrated numerically for given initial conditions at $z = 0$.

Finally, integrating Eqs. (35)–(37) and expressing the result in terms of redshift, the directional scale factors can be written as

$$A(z) = \frac{d_1}{1+z}\,\exp\left\{-\frac{(k_1+k_3)1G_0}{H_0}\int\frac{g(1+z)^2}{h}\,dz\right\}\ , \tag{56}$$

$$B(z) = \frac{d_2}{1+z}\,\exp\left\{\frac{(2k_3-k_1)G_0}{H_0}\int\frac{g(1+z)^2}{h}\,dz\right\}\ , \tag{57}$$

$$C(z) = \frac{d_3}{1+z}\,\exp\left\{\frac{(2k_1-k_3)G_0}{H_0}\int\frac{g(1+z)^2}{h}\,dz\right\}\ , \tag{58}$$

which shows explicitly how the anisotropic expansion histories in the x-, y- and z-directions are governed by the interplay between the mean expansion $h(z)$ and the running gravitational coupling $g(z)$. These expressions will be used later when discussing the evolution of anisotropy and the approach to isotropy at late times.

The deceleration parameter provides a quantitative measure of the expansion dynamics of the Universe. Its sign determines the nature of cosmic evolution: $q < 0$ signifies accelerated expansion, $q > 0$ denotes deceleration, and $q = 0$ marks the transition between decelerated and accelerated expansion. It is expressed as

$$q(z) = -1 - \frac{\dot{H}}{H^2} = -1 + (1+z)\frac{h'}{h}\ . \tag{59}$$

Using Eq. (43), this becomes

$$q(z) = -1 + \frac{3}{2}(1+w_m)\Omega_m + \frac{3}{2}(1+w_r)\Omega_r + 3\Omega_\sigma\ . \tag{60}$$

Furthermore, the deceleration parameter is related to the effective equation-of-state (EoS) parameter through

$$w_{\rm eff}(z) = \frac{2q(z)-1}{3}\ , \tag{61}$$

which characterize the effective expansion dynamics of the total cosmic fluid.

5 Results and Discussion

The anisotropic Bianchi type-I cosmological model in SD gravity is governed by a system of coupled, first-order nonlinear differential equations, Eqs. (50)–(55). These equations are solved numerically using a fourth-order Runge–Kutta method, with initial conditions $h(0) = 1, g(0) = 1, g'(0) = Z_0 = -0.037/H_0$, and $\Omega_{m0} = 0.269$, the latter taken from the Union2 supernova compilation. The resulting redshift evolution of the dynamical variables $h(z)$, θ, $\Omega_m(z)$, $\Omega_r(z)$, $\Omega_\sigma(z)$, $g(z)$, together with the cosmographic parameters $q(z)$, $w_{\rm eff}(z)$, is represented in Figures 1–8.

Figures (1) and (2) illustrate the evolution of the normalized Hubble parameter $h = H/H_0$ together with the expansion scalar $\theta = 3H$ in SD gravity. Both quantities evolve smoothly towards their expected present-day values, with the normalized parameter $h(0) = 1$ at $z = 0$. At early times, the expansion rate is enhanced due to the contribution of radiation and the non-trivial running of the gravitational coupling G_k. Unlike the standard ΛCDM model, the expansion history is determined not only by the matter and radiation components but also by the scale dependence of the gravitational coupling. The running of G_k modifies the effective strength of gravity throughout the cosmic evolution, while the anisotropic shear contributes an additional geometric source during the early Universe. Their combined effect alters the expansion rate without introducing extra matter fields or exotic dark-energy components.

As the Universe evolves, θ and h gradually settle into the late-time accelerating regime. Similarly, Figures (3) and (4) show the redshift evolution of the anisotropy and radiation density parameters, $\Omega_\sigma(z)$ and $\Omega_r(z)$, together with the normalized running gravitational coupling $g(z) = G(z)/G_0$. The anisotropy density parameter decays rapidly toward low redshift, indicating that the Bianchi type$-I$ Universe isotropizes during the cosmic expansion, consistent with CMB constraints on cosmic shear. This rapid decay suppresses the influence of anisotropy on the late-time dynamics, leaving the running gravitational coupling as the dominant source of deviations from the standard cosmological evolution. Consequently, the model naturally interpolates between an anisotropic early Universe and an effectively isotropic accelerating Universe.

The radiation density parameter dominates only at early times and becomes negligible at low redshift, as expected. At the present epoch, the running gravitational coupling satisfies $g(0) = 1$, ensuring consistency with local measurements of Newton's constant. Since g(z) is close to unity, the running of G_k is weak and the deviations from the ΛCDM model are expected to be small, but could be detected with future precision data. The weak evolution of $g(z)$ demonstrates that the scale dependence acts as a perturbative correction rather than a radical modification of GR. Nevertheless, even these small departures accumulate over cosmological timescales and produce measurable changes in the expansion history and matter evolution.

Figure (5) shows the redshift evolution of the matter density parameter $\Omega_m(z)$ within the SD gravity framework, together with the corresponding ΛCDM prediction. As shown, $\Omega_m(z)$ increases monotonically with redshift and approaches unity at high z, indicating a matter-dominated Universe in the past. The scale-dependent prediction (solid red curve) remains systematically below the ΛCDM result (black dashed curve) at all redshifts, reflecting a reduced effective matter fraction induced by the running gravitational coupling G_k and the cosmological term Λ_k. This deviation indicates that the matter sector does not evolve independently, but instead interacts with the gravitational vacuum through the scale dependence of the couplings. In contrast to ΛCDM, where the matter density evolves solely through the cosmic expansion, the SD framework introduces an indirect coupling between matter and geometry via the running gravitational parameters. As a result, the expansion history is modified while preserving the standard sequence of radiation, matter, and accelerated eras. On the other hand, the effective running-vacuum density parameter, $\Omega_\lambda(z) = 1-\Omega_m(z)-\Omega_r(z)$ attains its maximum value at $z = 0$ and decreases rapidly with increasing redshift, becoming negligible at early times. This behavior confirms that the vacuum sector is dynamically suppressed in the past and becomes relevant only at late times, where it drives the accelerated expansion. Overall, the evolution of $\Omega_m(z)$, together with the effective constraint $\Omega_\lambda(z)$ shows that the scale-dependent framework preserves the standard cosmological sequence of matter domination followed by late-time acceleration, while allowing for controlled deviations from ΛCDM arising from the running of the gravitational couplings.

Figures 7 and 8 illustrate the redshift evolution of the deceleration parameter $q(z)$ together with the effective equation of state (EoS) parameter $w_{\rm eff}(z)$ in the SD gravity framework. At the present epoch, the model predict a negative value of the deceleration parameter $q \sim -0.60$, which is in good agreement with observational constraints from Type Ia supernovae, cosmic microwave background measurements from *Planck*, and baryon acoustic oscillation data. The corresponding present-day effective EoS is $w_{\rm eff}(0) \sim -0.70$. As the redshift increases, $q(z)$ grows and crosses zero at the redshift transition $z_t \simeq 0.76$, signaling the transition from the current accelerated phase to the preceding decelerated era dominated by matter. This transition is also evident in the evolution of $w_{\rm eff}(z)$, which approaches the critical value $w_{\rm eff} = -1/3$ near the same redshift, as expected at the onset of cosmic acceleration. The transition is regulated by the combined evolution of the running gravitational coupling G_k, the effective cosmological term Λ_k, and the decaying anisotropic shear. While the shear modifies the expansion primarily during the early Universe, its contribution rapidly becomes negligible, allowing the SD gravitational couplings to govern the late-time acceleration. This interplay between anisotropy and running couplings constitutes one of the characteristic features of the present SD Bianchi type$-I$ model.

In the high-redshift regime ($z \gtrsim 2$), both $q(z)$ and $w_{\rm eff}(z)$ approach their matter-dominated limits, with $q \to \frac{1}{2}$ and $w_{\rm eff} \to 0$, confirming that the SD gravity model successfully recovers the standard matter-dominated expansion

history of the early Universe. As the redshift decreases, departures from these asymptotic values become apparent owing to the scale dependence of G_k and Λ_k, driving the cosmological evolution towards the late-time accelerating phase discussed above.

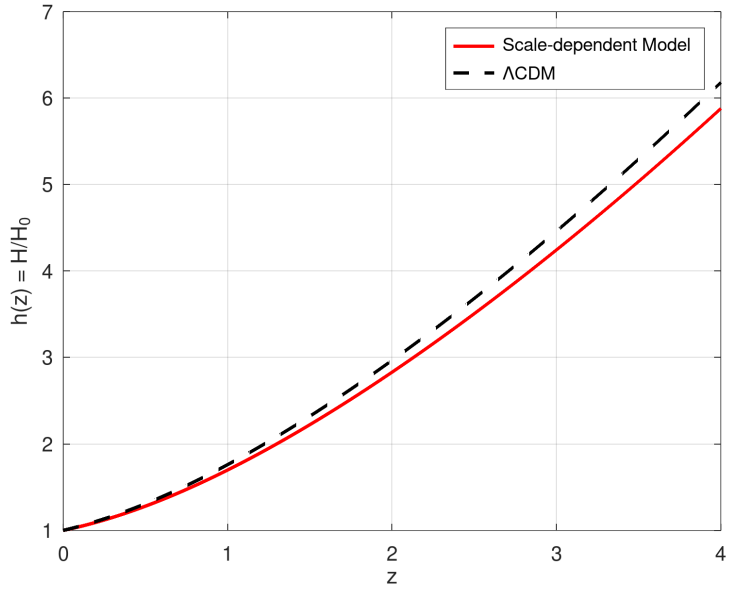


Figure 1: Redshift evolution of the normalized Hubble parameter $h(z)$ for the Bianchi type-I cosmological model within the framework of SD gravity. The initial conditions $h(0) = 1$, $Z_0 = g'(0) = -0.037/H_0$, $\Omega_{m0} = 0.269$ from Union2 compilation, $g(0) = 1$ are used to numerically integrate the DEs.

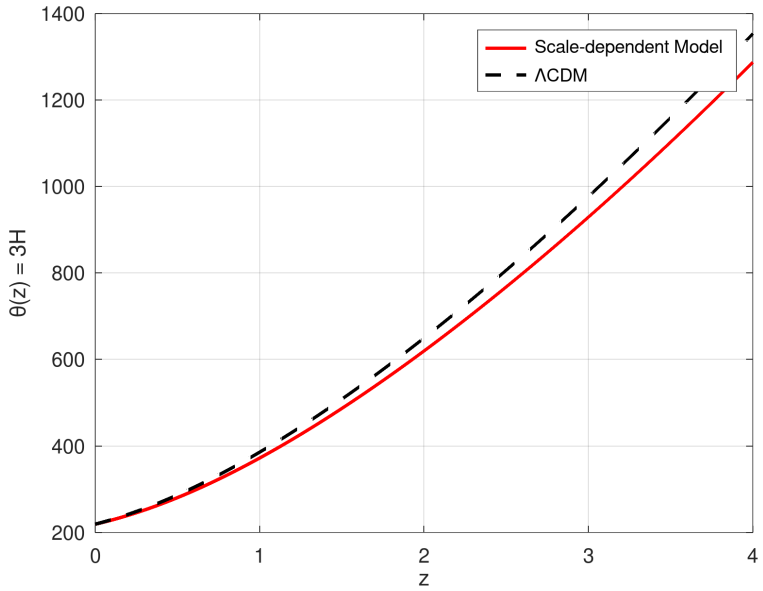


Figure 2: Redshift evolution of the expansion scalar $\theta(z)$ for the Bianchi type$-I$ cosmological model within the framework of SD gravity. The initial conditions $h(0) = 1$, $Z_0 = g'(0) = -0.037/H_0$, $\Omega_{m0} = 0.269$ from Union2 compilation, $g(0) = 1$ are used to numerically integrate the DEs.

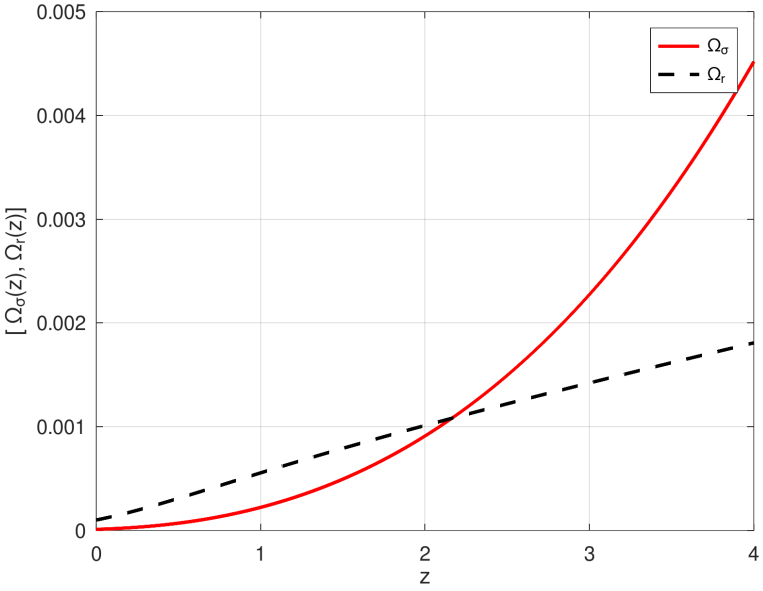


Figure 3: Redshift evolution of the shear and radiation density parameters, $\Omega_\sigma(z)$ and $\Omega_r(z)$, for the Bianchi type$-I$ cosmological model within the framework of SD gravity. The initial conditions $h(0) = 1$, $Z_0 = g'(0) = -0.037/H_0$, $\Omega_{m0} = 0.269$ from Union2 compilation, $g(0) = 1$ are used to numerically integrate the DEs.

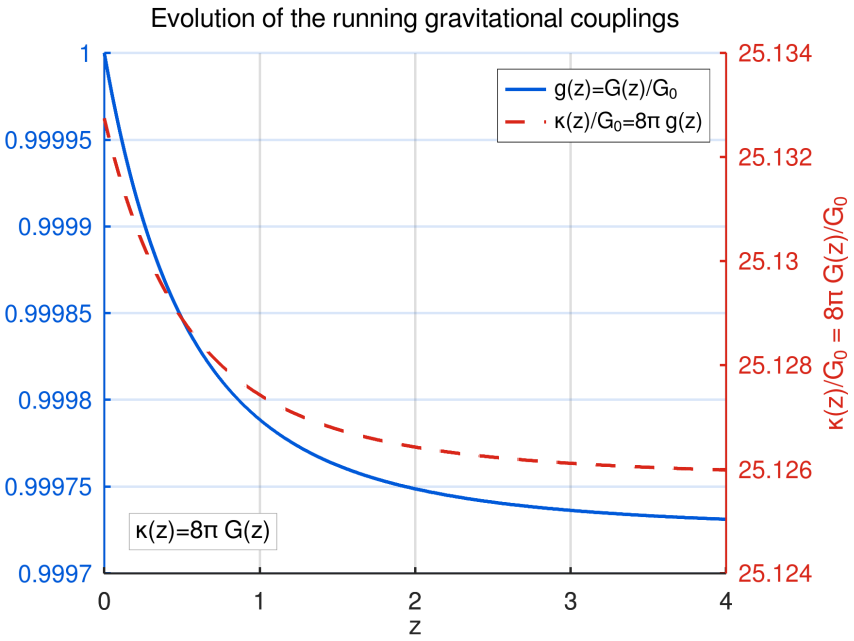


Figure 4: Redshift evolution of the running Newton coupling $g(z)$ and $\kappa(z)/G_0$ for the Bianchi type$-I$ cosmological model within the framework of SD gravity. The initial conditions $h(0) = 1$, $Z_0 = g'(0) = -0.037/H_0$, $\Omega_{m0} = 0.269$ from Union2 compilation, $g(0) = 1$ are used to numerically integrate the DEs.

6 Cosmographic parameters

In this section, we introduce additional diagnostic tests, including the state-finder parameters and the Om(z) diagnostic, to examine the evolutionary stages of cosmic acceleration and to identify the type of dark-energy responsible for the observed accelerated expansion of the Universe.

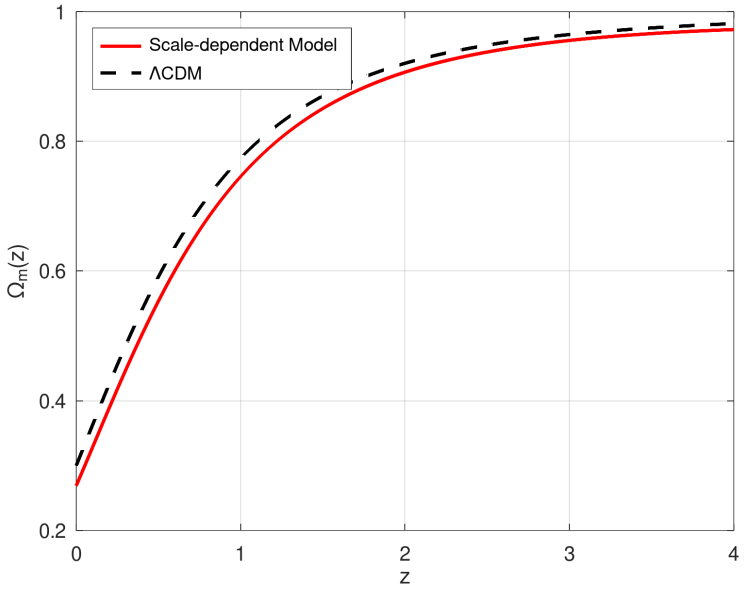


Figure 5: Redshift evolution of the matter density parameter $\Omega_m(z)$ for the Bianchi type-I cosmological model within the framework of SD gravity. The initial conditions $h(0) = 1$, $Z_0 = g'(0) = -0.037/H_0$, $\Omega_{m0} = 0.269$ from Union2 compilation, $g(0) = 1$ are used to numerically integrate the DEs.

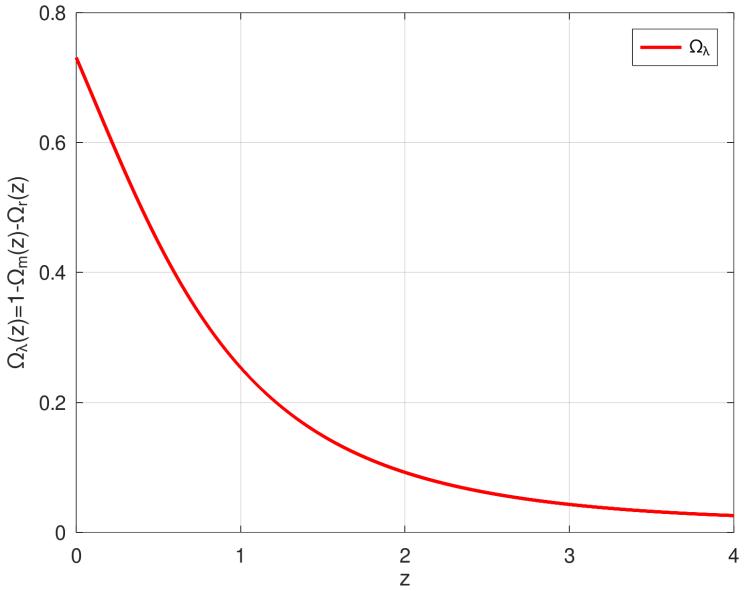


Figure 6: Redshift evolution of an effective running vacuum component $\Omega_\lambda(z) = 1 - \Omega_m(z) - \Omega_r(z)$ for the Bianchi type$-I$ cosmological model within the framework of scale-dependent gravity.

6.1 The state-finder parameters $r(z)$ and $s(z)$

The state-finder pair $\{r, s\}$ offers a geometric diagnostic capable of distinguishing dynamical dark-energy models from ΛCDM through the use of higher derivatives of the scale factor. The parameter

$$r(z) = 2q(z)^2 + q(z) + (1+z)\frac{\mathrm{d}q}{\mathrm{d}z}\,, \tag{62}$$

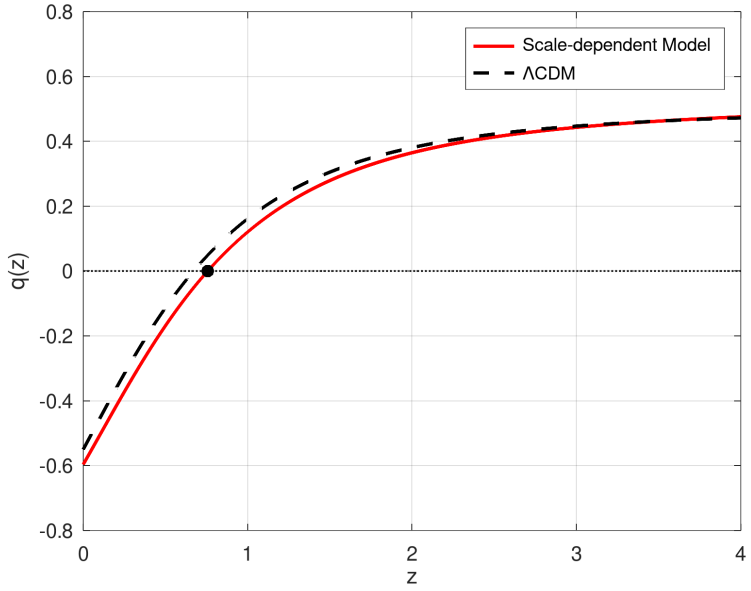


Figure 7: Redshift evolution of the deceleration parameter q for the Bianchi type-I cosmological model within the framework of SD gravity. The initial conditions $h(0) = 1$, $Z_0 = g'(0) = -0.037H_0$, $\Omega_{m0} = 0.269$ from Union2 compilation, $g(0) = 1$ are used to numerically integrate the DEs.

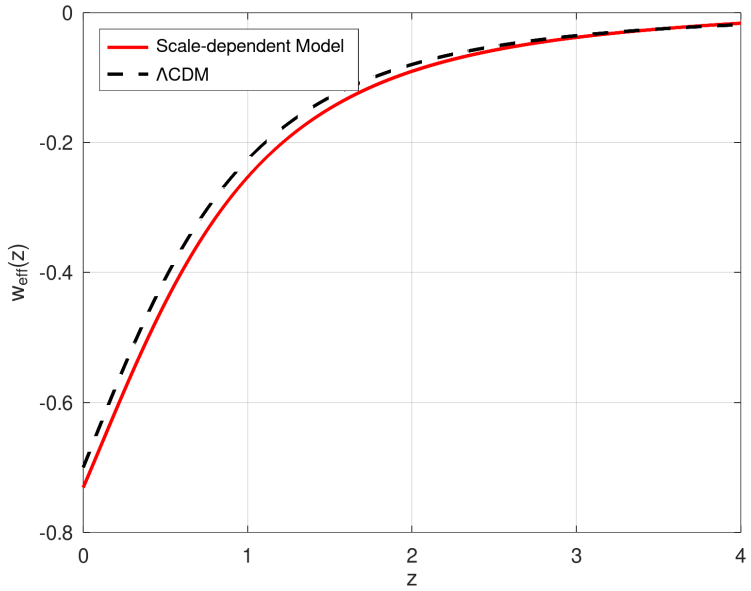


Figure 8: Redshift evolution of the effective equation of state $w_{\rm eff}$ for the Bianchi type-I cosmological model within the framework of SD gravity. The initial conditions $h(0) = 1$, $Z_0 = g'(0) = -0.037H_0$, $\Omega_{m0} = 0.269$ from Union2 compilation, $g(0) = 1$ are used to numerically integrate the DEs.

corresponds to the cosmic *jerk* and is particularly sensitive to variations in the deceleration parameter. The complementary quantity

$$s(z) = \frac{r(z) - 1}{3\left(q(z) - \frac{1}{2}\right)}\,, \tag{63}$$

isolates the dark-energy dynamics by minimizing explicit dependence on the matter content. The fixed point $(r, s) = (1, 0)$ characterizes ΛCDM. In the (q, r) plane, ΛCDM trajectory lies along $r = 1$, approaching the de Sitter point $(q, r) = (-1, 1)$. Within the scale-dependent gravity framework, the running couplings G_k and Λ_k induce smooth departures from these fixed points at low and intermediate redshifts. In the r–s plane, the trajectory

predominantly occupies the region $s < 0$ and $r > 1$, which is characteristic of quintessence-like behavior. In the r–q plane, the evolution interpolates smoothly between the matter-dominated era and the late-time accelerated phase, approaching the de Sitter limit asymptotically.

The convergence of the trajectories towards the ΛCDM fixed points demonstrates that the model reproduces the correct asymptotic expansion history, while the intermediate deviations provide a clear geometric signature of dynamical effects induced by scale-dependent gravitational couplings.

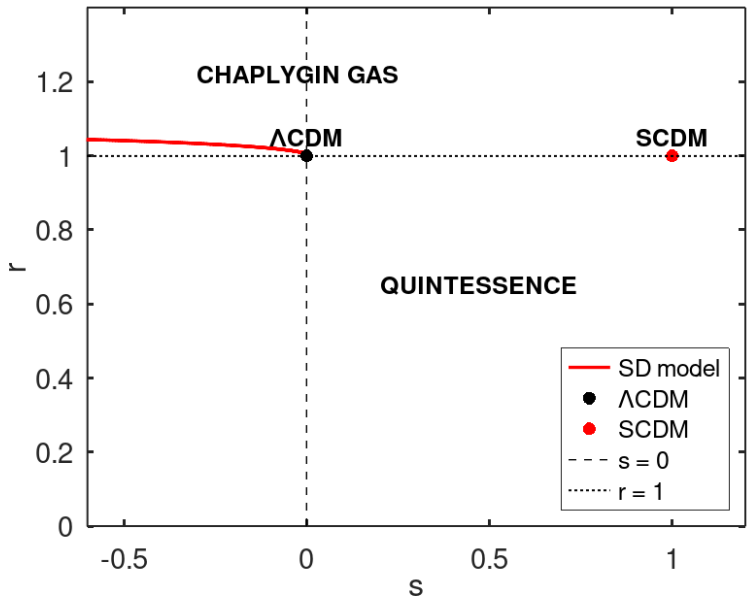


Figure 9: Statefinder diagnostic in the r–s plane for the scale-dependent gravity model. The trajectories depart smoothly from the ΛCDM fixed points and evolve predominantly in the quintessence-like region at low and intermediate redshifts. At late times, the trajectories tend toward the standard limits, indicating that the model reproduces the correct asymptotic expansion history while exhibiting controlled deviations induced by the running gravitational couplings.

6.2 The diagnostic test Om(z)

The $\mathrm{Om}(z)$ diagnostic, which was first introduced by [20], is an independent and reliable tool for distinguishing among alternative dark-energy theories and identifying the mechanism driving cosmic acceleration. It is defined by

$$\mathrm{Om}(z) = \frac{h^2 - 1}{(1+z)^3 - 1} \,. \tag{64}$$

In the conventional concordance model ΛCDM, $\mathrm{Om}(z)$ is a constant for all redshifts, i.e., the cosmological constant does not depend on the redshift. A positive or negative slope of the $\mathrm{Om}(z)$ diagnostic indicates phantom-like or quintessence-like behavior, respectively. Departs of $\mathrm{Om}(z)$ from constancy indicate deviations from ΛCDM and can be used to assess whether the cosmic acceleration arises from a true cosmological constant or a dynamical dark energy component.

Figure (11) shows that $\mathrm{Om}(z)$ increases mildly with redshift and therefore decreases as the Universe evolve toward the present epoch. This evolution departs from the constant ΛCDM prediction and is consistent with a quintessence-like effective behavior.

7 Conclusion

In this article, we have examined the cosmological implications of SD gravity in a spatially homogeneous but anisotropic Bianchi type$-I$ spacetime, allowing both the Newton coupling G_k and the cosmological term Λ_k to evolve dynamically with cosmic time t. The modified Einstein field equations (MEFEs) have been reduced to a

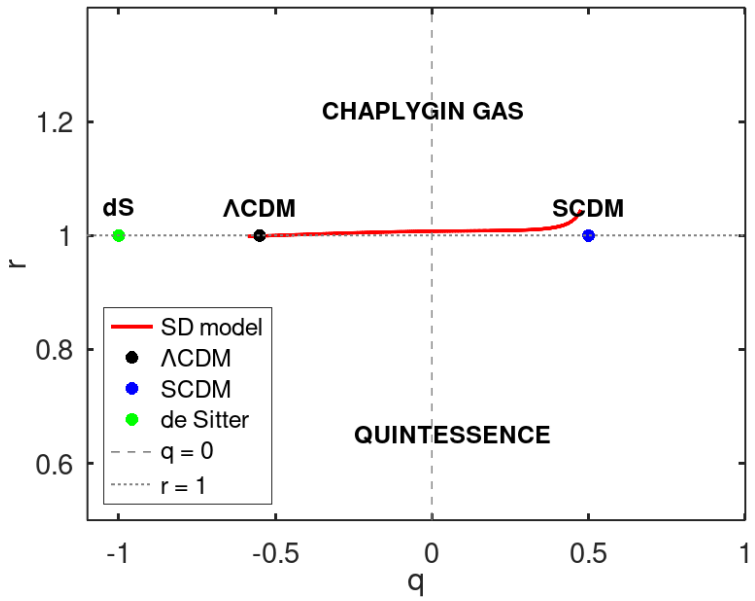


Figure 10: Statefinder diagnostic in the r–q plane for the scale-dependent gravity model. The trajectories depart smoothly from the ΛCDM fixed points and evolve predominantly in the quintessence-like region at low and intermediate redshifts. At late times, the trajectories tend toward the standard limits, indicating that the model reproduces the correct asymptotic expansion history while exhibiting controlled deviations induced by the running gravitational couplings.

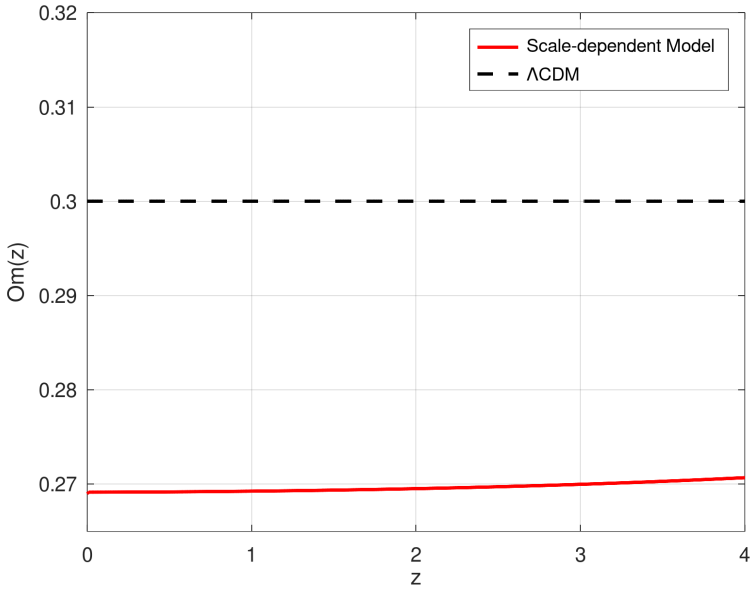


Figure 11: Redshift evolution of the Om(z) diagnostic for the SD model and ΛCDM.

closed set of dimensionless, first-order differential equations (DEs) in redshift space and solved numerically using observationally consistent initial conditions (ICs). The obtained results in §5 show that the scale-dependent framework successfully reproduces the standard cosmological sequence of radiation and matter domination, followed by a late-time accelerated expansion. The anisotropic shear contribution shown in Figure (3) decays rapidly with expansion, leading to an efficient isotropization of the Universe at low redshift, in agreement with observational constraints from the cosmic microwave background (CMB). At the present epoch $z = 0$, the model yields a negative deceleration parameter $q_0 \approx -0.60$ and an effective equation of state (EoS) $w_{\text{eff}} \approx -0.70$, qualitatively consistent with quintessence-like dark-energy behavior. The obtained results are compatible with current obser-

vational constraints. Moreover, this behavior is further supported by the diagnostic analysis of the state-finder parameter, where the trajectories in the (r, s) and (r, q) planes smoothly depart from the ΛCDM fixed point at low redshift z and evolve predominantly within the quintessence region, while asymptotically reaching the standard limits. Also, the negative slope in Fig. (11) of the $\mathrm{Om}(z)$ diagnostic indicates that the late-time acceleration of the Universe is caused by a quintessence-like (dynamical dark-energy) rather than a strict cosmological constant Λ. Our results indicate that SD gravity constitute a consistent and viable extension of standard cosmology, providing a possible framework for describing late-time cosmic acceleration while preserving the main features of ΛCDM at early and late times. The inclusion of anisotropy improves the SD framework beyond the isotropic limits and provides a broader setting in which to evaluate such modified gravity scenarios.

Future work may constrain the model using Type Ia supernovae (SNe Ia), baryon acoustic oscillations (BAO), cosmic microwave background (CMB) data, and cosmic chronometer measurements (CC) measurements, and may investigate cosmological perturbations, structure formation, and other anisotropic cosmological models.

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A Derivation of the Directional Metric Functions

Although the background cosmological evolution is completely determined by the averaged field equations and the effective Raychaudhuri equation (27), it is useful to reconstruct the individual directional scale factors in order to describe the anisotropic geometry explicitly. In particular, the functions $A(t)$, $B(t)$ and $C(t)$ determine the directional Hubble expansion rates and the shear scalar, thereby characterizing the departure from isotropy. While these quantities are not directly observable, they provide the geometrical foundation of the Bianchi type-I spacetime and demonstrate how the running gravitational coupling modifies the directional expansion.

Starting from the spatial components of the modified field equations, the isotropic contributions associated with the effective pressure cancel upon taking pairwise differences. For example, subtracting the (22) equation from the (11) equation yields

$$\frac{d}{dt}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)+\left(3H+\frac{\dot{G}_k}{G_k}\right)\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=0, \tag{65}$$

where

$$H=\frac{1}{3}\left(H_x+H_y+H_z\right), \qquad H_x=\frac{\dot{A}}{A}, \quad H_y=\frac{\dot{B}}{B}, \quad H_z=\frac{\dot{C}}{C}.$$

Similarly, subtracting the remaining pairs of spatial equations gives

$$\frac{d}{dt}\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)+\left(3H+\frac{\dot{G}_k}{G_k}\right)\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)=0, \tag{66}$$

$$\frac{d}{dt}\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)+\left(3H+\frac{\dot{G}_k}{G_k}\right)\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)=0. \tag{67}$$

Since

$$3H=\frac{\dot{V}}{V}, \qquad V=ABC=a^3,$$

these equations can be integrated directly to give

$$\frac{\dot{C}}{C}-\frac{\dot{A}}{A}=\frac{k_1 G_k}{a^3}, \tag{68}$$

$$\frac{\dot{C}}{C}-\frac{\dot{B}}{B}=\frac{k_2 G_k}{a^3}, \tag{69}$$

$$\frac{\dot{B}}{B}-\frac{\dot{A}}{A}=\frac{k_3 G_k}{a^3}, \tag{70}$$

where k_1, k_2, and k_3 are integration constants satisfying the consistency relation

$$k_3=k_1-k_2. \tag{71}$$

A second integration immediately yields the ratios of the directional scale factors,

$$\frac{C}{A}=m_1\exp\left(k_1\int\frac{G_k}{a^3}\,dt\right), \tag{72}$$

$$\frac{C}{B}=m_2\exp\left(k_2\int\frac{G_k}{a^3}\,dt\right), \tag{73}$$

$$\frac{B}{A}=m_3\exp\left(k_3\int\frac{G_k}{a^3}\,dt\right), \tag{74}$$

where m_1, m_2, and m_3 are integration constants with

$$m_3 = \frac{m_1}{m_2}.$$

Finally, using the volume constraint

$$ABC = a^3,$$

the directional metric functions can be reconstructed as

$$A = d_1 a \exp\left[-\frac{k_1 + k_3}{3} \int \frac{G_k}{a^3}\, dt\right], \tag{75}$$

$$B = d_2 a \exp\left[\frac{2k_3 - k_1}{3} \int \frac{G_k}{a^3}\, dt\right], \tag{76}$$

$$C = d_3 a \exp\left[\frac{2k_1 - k_3}{3} \int \frac{G_k}{a^3}\, dt\right], \tag{77}$$

where the constants d_i satisfy

$$d_1 d_2 d_3 = 1,$$

together with the relations given in the main text.

Differentiating the above expressions leads directly to the directional Hubble parameters,

$$H_x = \frac{\dot{A}}{A}, \tag{78}$$

$$H_y = \frac{\dot{B}}{B}, \tag{79}$$

$$H_z = \frac{\dot{C}}{C}, \tag{80}$$

from which Eqs. (50)–(52) follow immediately. It should be emphasized that these relations are reconstruction equations rather than independent dynamical equations. They do not affect the evolution of the averaged background quantities governed by Eq. (27), but instead recover the complete anisotropic geometry once the average scale factor $a(t)$ and the running coupling $G_k(t)$ are known. Consequently, the present appendix serves only to reconstruct the directional expansion and to demonstrate explicitly how the scale dependence of the gravitational coupling is encoded in the anisotropic sector.

B Explicit Derivation of the Redshift-Space Equations

In this appendix, we provide the explicit derivation of the redshift-space evolution equations for the dimensionless variables

$$h(z) = \frac{H(z)}{H_0}, \qquad g(z) = \frac{G_k(z)}{G_0}, \qquad Z(z) = g'(z),$$

together with the density parameters

$$\Omega_m = \frac{\kappa G_k \rho_m}{3H^2}, \qquad \Omega_r = \frac{\kappa G_k \rho_r}{3H^2}, \qquad \Omega_\sigma = \frac{\sigma^2}{3H^2}\,, \qquad \Omega_\lambda = \frac{\Lambda_k}{3H^2} + \Omega_\sigma + \frac{\dot{G}_k}{HG_k}.$$

Here $\kappa = 8\pi$. Throughout this appendix, a dot denotes differentiation with respect to cosmic time t, whereas a prime denotes differentiation with respect to redshift z.

B.1 Transformation between cosmic time and redshift

Since

$$a = \frac{1}{1+z},$$

we have

$$\frac{dz}{dt} = -(1+z)H.$$

Therefore, for any differentiable quantity $Q(z)$,

$$\dot{Q} = \frac{dQ}{dz}\frac{dz}{dt} = -(1+z)HQ'. \tag{81}$$

Differentiating once more gives

$$\begin{aligned}\ddot{Q} &= \frac{d}{dt}\left[-(1+z)HQ'\right] \\ &= -\frac{d(1+z)}{dt}HQ' - (1+z)\dot{H}Q' - (1+z)H\frac{dQ'}{dt}.\end{aligned} \tag{82}$$

Using

$$\frac{d(1+z)}{dt} = -(1+z)H$$

and

$$\frac{dQ'}{dt} = -(1+z)HQ'',$$

we obtain

$$\ddot{Q} = (1+z)^2H^2Q'' + (1+z)^2HH'Q' + (1+z)H^2Q'. \tag{83}$$

For $H = H_0h$, Eq. (81) gives

$$\dot{H} = -(1+z)HH' = -(1+z)H^2\frac{h'}{h}, \tag{84}$$

and hence

$$\frac{\dot{H}}{H^2} = -(1+z)\frac{h'}{h}. \tag{85}$$

Similarly, since $G_k = G_0g$,

$$\dot{G}_k = -(1+z)HG_0g', \tag{86}$$

so that

$$\frac{\dot{G}_k}{G_k} = -(1+z)H\frac{g'}{g} = -(1+z)H\frac{Z}{g}. \tag{87}$$

B.2 Derivation of the normalized Hubble equation

The Raychaudhuri equation used in the manuscript is

$$\dot{H} = -\frac{\kappa G_k}{2}\left[(\rho_m + p_m) + (\rho_r + p_r)\right] - \sigma^2. \tag{88}$$

Introducing the equations of state

$$p_m = w_m\rho_m, \qquad p_r = w_r\rho_r,$$

Eq. (88) becomes

$$\dot{H} = -\frac{\kappa G_k}{2}\left[(1+w_m)\rho_m + (1+w_r)\rho_r\right] - \sigma^2. \tag{89}$$

Using

$$\frac{\kappa G_k\rho_m}{3H^2} = \Omega_m, \qquad \frac{\kappa G_k\rho_r}{3H^2} = \Omega_r, \qquad \frac{\sigma^2}{3H^2} = \Omega_\sigma,$$

we find

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}(1+w_m)\Omega_m - \frac{3}{2}(1+w_r)\Omega_r - 3\Omega_\sigma. \tag{90}$$

On the other hand, Eq. (85) gives

$$\frac{\dot{H}}{H^2} = -(1+z)\frac{h'}{h}.$$

Equating the two expressions yields

$$(1+z)\frac{h'}{h} = \frac{3}{2}(1+w_m)\Omega_m + \frac{3}{2}(1+w_r)\Omega_r + 3\Omega_\sigma. \tag{91}$$

Therefore,

$$\boxed{h' = \frac{h}{1+z}\left[\frac{3}{2}(1+w_m)\Omega_m + \frac{3}{2}(1+w_r)\Omega_r + 3\Omega_\sigma\right].} \tag{92}$$

Thus, the factor 3 multiplying Ω_σ is not an additional or spurious factor. It follows directly from the definition

$$\Omega_\sigma = \frac{\sigma^2}{3H^2}.$$

B.3 Evolution equation for the matter density parameter

Starting from

$$\Omega_m = \frac{\kappa G_k \rho_m}{3H^2},$$

its logarithmic derivative is

$$\frac{\dot{\Omega}_m}{\Omega_m} = \frac{\dot{G}_k}{G_k} + \frac{\dot{\rho}_m}{\rho_m} - 2\frac{\dot{H}}{H}. \tag{93}$$

The matter conservation equation is

$$\dot{\rho}_m + 3H(1+w_m)\rho_m = 0, \tag{94}$$

and therefore

$$\frac{\dot{\rho}_m}{\rho_m} = -3H(1+w_m). \tag{95}$$

Substitution into Eq. (93) gives

$$\dot{\Omega}_m = \Omega_m\left[\frac{\dot{G}_k}{G_k} - 3H(1+w_m) - 2\frac{\dot{H}}{H}\right]. \tag{96}$$

Using

$$\dot{\Omega}_m = -(1+z)H\Omega_m',$$

together with Eqs. (87) and (85), we obtain

$$\boxed{\Omega_m' = \left[-2\frac{h'}{h} + \frac{Z}{g} + \frac{3(1+w_m)}{1+z}\right]\Omega_m.} \tag{97}$$

B.4 Evolution equation for the radiation density parameter

Similarly,

$$\Omega_r = \frac{\kappa G_k \rho_r}{3H^2},$$

and

$$\frac{\dot{\Omega}_r}{\Omega_r} = \frac{\dot{G}_k}{G_k} + \frac{\dot{\rho}_r}{\rho_r} - 2\frac{\dot{H}}{H}. \tag{98}$$

For radiation,

$$\dot{\rho}_r + 3H(1+w_r)\rho_r = 0. \tag{99}$$

Hence,

$$\boxed{\Omega_r' = \left[-2\frac{h'}{h} + \frac{Z}{g} + \frac{3(1+w_r)}{1+z}\right]\Omega_r.} \tag{100}$$

For $w_r = 1/3$, this reduces to

$$\boxed{\Omega_r' = \left[-2\frac{h'}{h} + \frac{Z}{g} + \frac{4}{1+z}\right]\Omega_r.} \tag{101}$$

B.5 Reconstruction of the effective density parameter

In the present formulation, the effective vacuum density parameter is defined by

$$\Omega_\lambda = \frac{\Lambda_k}{3H^2} + \Omega_\sigma + \frac{\dot{G}_k}{HG_k}. \tag{102}$$

Using Eq. (87), the last contribution becomes

$$\frac{\dot{G}_k}{HG_k} = -(1+z)\frac{Z}{g}, \tag{103}$$

so that

$$\boxed{\Omega_\lambda = \Omega_\Lambda + \Omega_\sigma - (1+z)\frac{Z}{g}.} \tag{104}$$

Since Ω_λ is determined algebraically through the modified Friedmann constraint,

$$\Omega_m + \Omega_r + \Omega_\lambda = 1, \tag{105}$$

no independent evolution equation for Ω_λ is required.

B.6 Evolution equation for the shear density parameter

The shear density parameter is

$$\Omega_\sigma = \frac{\sigma^2}{3H^2}. \tag{106}$$

Its logarithmic derivative is

$$\frac{\Omega_\sigma'}{\Omega_\sigma} = \frac{(\sigma^2)'}{\sigma^2} - 2\frac{h'}{h}. \tag{107}$$

At this point, two different shear laws must be distinguished.

B.6.1 Case I: standard shear law $\sigma \propto a^{-3}$

If one assumes

$$\sigma = \sigma_0 a^{-3}, \tag{108}$$

then

$$\sigma^2 = \sigma_0^2 a^{-6} = \sigma_0^2 (1+z)^6. \tag{109}$$

Therefore,

$$\frac{(\sigma^2)'}{\sigma^2} = \frac{6}{1+z}. \tag{110}$$

Equation (107) then yields

$$\boxed{\Omega_\sigma' = \Omega_\sigma \left[\frac{6}{1+z} - 2\frac{h'}{h}\right].} \tag{111}$$

Under this assumption, no term proportional to Z/g appears.

B.6.2 Case II: shear obtained from the directional equations

The directional equations of the scale-dependent Bianchi I model give

$$H_i - H_j \propto \frac{G_k}{a^3}.$$

Since the shear scalar is constructed quadratically from the directional Hubble differences,

$$\sigma^2 \propto \frac{G_k^2}{a^6}. \tag{112}$$

Writing

$$\sigma^2 = \sigma_0^2 g^2 (1+z)^6,$$

we obtain

$$\frac{(\sigma^2)'}{\sigma^2} = 2\frac{g'}{g} + \frac{6}{1+z} = 2\frac{Z}{g} + \frac{6}{1+z}. \tag{113}$$

Equation (107) then gives

$$\boxed{\Omega_\sigma' = \Omega_\sigma \left[\frac{6}{1+z} + 2\frac{Z}{g} - 2\frac{h'}{h}\right].} \tag{114}$$

Therefore, if the shear is derived from the directional field equations, the Z/g contribution enters with a *positive* sign. Throughout this work the shear evolution is taken from the directional equations of the scale-dependent Bianchi I model, Eq. (114).

B.7 Evolution equation for $Z = g'$

We now derive the second-order equation for $g(z)$ from the null condition

$$\frac{\ddot{G}_k}{G_k} - 2\left(\frac{\dot{G}_k}{G_k}\right)^2 - H\frac{\dot{G}_k}{G_k} = 0. \tag{115}$$

Since $G_k = G_0 g$, Eq. (83) gives

$$\ddot{G}_k = G_0\left[(1+z)^2 H^2 g'' + (1+z)^2 H H' g' + (1+z) H^2 g'\right]. \tag{116}$$

Therefore,

$$\frac{\ddot{G}_k}{G_k} = (1+z)^2 H^2 \frac{g''}{g} + (1+z)^2 H H' \frac{g'}{g} + (1+z) H^2 \frac{g'}{g}. \tag{117}$$

Furthermore,

$$\left(\frac{\dot{G}_k}{G_k}\right)^2 = (1+z)^2 H^2 \left(\frac{g'}{g}\right)^2, \tag{118}$$

and

$$-H\frac{\dot{G}_k}{G_k} = (1+z) H^2 \frac{g'}{g}. \tag{119}$$

Substitution into Eq. (115) gives

$$0 = (1+z)^2 H^2 \frac{g''}{g} + (1+z)^2 H H' \frac{g'}{g} + 2(1+z) H^2 \frac{g'}{g} - 2(1+z)^2 H^2 \left(\frac{g'}{g}\right)^2 . \tag{120}$$

Dividing by $(1+z)^2 H^2$ yields

$$\frac{g''}{g} + \frac{H'}{H}\frac{g'}{g} + \frac{2}{1+z}\frac{g'}{g} - 2\left(\frac{g'}{g}\right)^2 = 0. \tag{121}$$

Since

$$\frac{H'}{H} = \frac{h'}{h},$$

we obtain

$$g'' = 2\frac{g'^2}{g} - \left[\frac{2}{1+z} + \frac{h'}{h}\right] g'. \tag{122}$$

Finally, introducing

$$Z = g',$$

we have

$$\boxed{g' = Z,} \tag{123}$$

and

$$\boxed{Z' = 2\frac{Z^2}{g} - \left[\frac{2}{1+z} + \frac{h'}{h}\right] Z.} \tag{124}$$

which is the evolution equation for Z used in the main text, Eq. (55).

B.8 Redshift-space system

Using the shear evolution implied by the directional equations, the closed redshift-space system is

$$h' = \frac{h}{1+z}\left[\frac{3}{2}(1+w_m)\Omega_m + \frac{3}{2}(1+w_r)\Omega_r + 3\Omega_\sigma\right], \tag{125}$$

$$\Omega_m' = \left[-2\frac{h'}{h} + \frac{Z}{g} + \frac{3(1+w_m)}{1+z}\right]\Omega_m, \tag{126}$$

$$\Omega_r' = \left[-2\frac{h'}{h} + \frac{Z}{g} + \frac{3(1+w_r)}{1+z}\right]\Omega_r, \tag{127}$$

$$\Omega_\sigma' = \Omega_\sigma\left[\frac{6}{1+z} + 2\frac{Z}{g} - 2\frac{h'}{h}\right], \tag{128}$$

$$g' = Z, \tag{129}$$

$$Z' = 2\frac{Z^2}{g} - \left[\frac{2}{1+z} + \frac{h'}{h}\right] Z. \tag{130}$$

with

$$\Omega_\lambda = 1 - \Omega_m - \Omega_r.$$

If instead the standard phenomenological assumption $\sigma = \sigma_0 a^{-3}$ were retained independently of the directional equations, the shear equation would read

$$\Omega_\sigma' = \Omega_\sigma \left[\frac{6}{1+z} - 2\frac{h'}{h} \right].$$

This prescription is not used here: the directional-equation form, Eq. (114), is adopted consistently in both the analytical derivation and the numerical analysis.